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ScienceDirect
J. Math. Anal. Appl. 332 (2007) 631–665
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

The Kustaanheimo–Stiefel map, the Hopf fibration and the square root map on $\mathbb{R}^3$ and $\mathbb{R}^4$

Mohamed Sami ElBialy

Mathematics Department, University of Toledo, Toledo, OH 43606, USA

Received 11 January 2006

Available online 22 November 2006

Submitted by M.M. Peloso

Abstract

We study the Kustaanheimo–Stiefel map (KSM) ψ from $\mathbb{U}^* := \mathbb{R}^4 \setminus \{0\}$ to $\mathbb{X}^* := \mathbb{R}^3 \setminus \{0\}$ and the principal circle bundle $\mathcal{P} = (\mathbb{U}^*, \psi, \mathbb{X}^*, S^1)$ that it induces. We show that the KSM is the appropriate generalization of the squaring map $z \mapsto z^2$, $z \in \mathbb{C}$, and not quaternion-multiplication, in that the KSM induces a principal circle bundle on $S^3 \to S^2$, namely the Hopf fibration, while quaternion-squaring is degenerate because the dimension of the fibers is not constant.

We construct two square root branches from the upper and lower half of $\mathbb{R}^3$ to $\mathbb{R}^3 \setminus (x_1)^-$ where $(x_1)^-$ is the nonpositive x_1-axis in $\mathbb{R}^3$ and resembles the cut used to define the standard complex square root branches $\pm\sqrt{z}$. We glue these two branches together.

We introduce what we like to call *KS cylindrical coordinates* with a 2-dimensional axis of rotation. We also introduce what we call *KS torical and spherical coordinates*.

We use the *KS cylindrical coordinates* to define the full square root map on an S^1-cover of $\mathbb{R}^3$ given by $(\mathbb{R}^3 \times S^1)/\sim$, where $\sim$ is an equivalence relation on $(x_1)^- \times S^1$.

Keywords: Hopf fibration; Hopf map; Kustaanheimo–Stiefel transformation; Levi-Civita transformation; Hurwitz theorem; Collision singularities; Hypercomplex structures; Quaternionic structures; Circle bundles

1. Introduction

In 1920 Levi-Civita [11] used what came to be known as the *Levi-Civita transformation* to regularize the planar perturbed Kepler problem

E-mail address: melbialy@math.utoledo.edu.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2006.09.079

$$\begin{aligned}
&\dot{x} = y,\\
&\dot{y} = -\frac{a}{|x|^3}x + f,\\
&\dot{\alpha} = y^{\mathsf{T}} f, \qquad \alpha = \frac{|y|^2}{2} - \frac{a}{|x|}.
\end{aligned} \tag{1.1}$$

His method is geometric and amounts to a double cover $x = z^2$ of the complex plane. In matrix notation the Levi-Civita transformation (LCT) is given by

$$\begin{aligned}
&(u, w, \tau) \mapsto (x, y, t),\\
&x = \psi(u) = L(u)u,\\
&y = |u|^{-2} L(u)w,\\
&\frac{d}{d\tau} = |u|^2 \frac{d}{dt},\\
&L(u) = \begin{pmatrix} u_1 & -u_2 \\ u_2 & u_1 \end{pmatrix} = \begin{pmatrix} u^{(1)} & u^{(2)} \end{pmatrix},\\
&x = (x_1, x_2)^{\mathsf{T}}, \qquad u = (u_1, u_2)^{\mathsf{T}}, \ldots.
\end{aligned} \tag{1.2}$$

The Levi-Civita transformation sends the perturbed Kepler system (1.1) to the perturbed harmonic oscillator

$$\begin{aligned}
&u' = \frac{1}{2} w,\\
&w' = \alpha u + |u|^2 L(u)^{\mathsf{T}} f,\\
&\alpha' = f^{\mathsf{T}} L(u) w, \qquad \alpha |u|^2 = \frac{1}{2}|w|^2 - a.
\end{aligned} \tag{1.3}$$

In complex numbers notation $x = \psi(u) = u^2$ and $y = w/\bar{u}$. Locally the *Levi-Civita map* (LCM) $\psi(u)$ is *real bi-analytic*, that is, a real analytic bijection with a real analytic inverse.

The matrix $L(u)$ is (a) orthogonal, (b) linear in u, and (c) its first column is u. The columns of $L(u)$, $\{u^{(1)}, u^{(2)}\}$, provide a real analytic orthogonal frame for $\mathbb{R}^2 \setminus \{0\}$. In other words, the unit circle S^1 has a real analytic nonvanishing vector field, namely $u^{(2)}$, which means that S^1 is parallelizable.[1] An even-dimensional sphere does not possess a single nonvanishing continuous vector field. Odd-dimensional spheres do. Among odd-dimensional spheres, only S^1, S^3 and S^7 are parallelizable [2,8]. This is equivalent to a celebrated theorem by A. Hurwitz [7] which says that square matrices that satisfy (a)–(c) can only be of size 1, 2, 4, and 8.

Levi-Civita tried to generalize his regularization technique to the three-dimensional Kepler problem but without any success because of Hurwitz's theorem. "*This may be the reason his ingenious method is not described in most of the textbooks of celestial mechanics*" [14, p. 23].

P. Kustaanheimo and E. Stiefel [9,10,14] used their expertise in the theory of spinors and topology to extend the Levi-Civita transformation to the three-dimensional Kepler problem by introducing the *Kustaanheimo–Stiefel transformation* (KST). In fact if we let $u \in \mathbb{R}^4$ and replace $L(u)$ in (1.2) by $L(u)$ given in (1.4), then $x = L(u)u$ is in $\mathbb{R}^3 \times \{0\}$ and we obtain the KST and the regularized vector field (1.3). Of course there is more to the KST than that. But the point is that the LCT and the KST take the same form but in different dimensions.

[1] An n-dimensional manifold is *parallelizable* iff it has n nonvanishing continuous vector fields.

1.1. The Hopf map and the squaring map. When restricted to the unit sphere, both the Levi-Civita map and the KS map, $u \mapsto x = L(u)u$, are special cases of the Hopf map $H_{i,j} : S^i \to S^j$, $(i, j) = (1, 1), (3, 2), (7, 4)$ [12, §20]. We consider these maps as maps on Euclidean spaces and write $\mathcal{L}_{n,k} : \mathbb{R}^n \to \mathbb{R}^k$, $(n, k) = (2, 2), (4, 3), (8, 8)$. But $\mathcal{L}_{2,2}$ is the Levi-Civita map which is the standard squaring of complex numbers. And we know how to define a square root map on a two sheeted Riemann surface. We will see soon that we can view the KS map as a squaring map (in fact the upper left corner of (1.4) is the LC map). Thus, we should be able to construct what we might reasonably call a square root map that corresponds to the KS map. But in this case it should be made of 3-dimensional manifolds and not sheets and it should be an S^1-cover rather than a double cover. In fact $H_{3,2}$, equivalently $\mathcal{L}_{4,3}$, define a circle bundle.

1.2. In this work we focus on the geometry of the KS map. We construct a square root map and define what we call *KS cylindrical, torical* and *spherical coordinates*.

1.3. In [4] we use parts of our present work to generalize our work in [3] to simultaneous binary collision (SBC) singularities in $\mathbb{R}^3$. In [4] we show that SB collision and ejection orbits can be *collectively analytically continued*. That is, can be written as a convergent power series in $\tau = t^{1/3}$ with coefficients that depend real analytically on initial conditions that lie in a real analytic submanifold. We demonstrate this fact geometrically without using any power series techniques nor relying on any assertions that are demonstrated using power series techniques.

We introduce a KS transformation for each binary. We use the intrinsic energies as variables as we did above. Then we use the KS multiplication $(u_1, u_2) \mapsto L(u_1)u_2$. This is not associative but recall that although quaternion multiplication is associative, multiplication of octonions is not. In the complex plane the direction of z_2 relative to z_1 is given by $z_2/z_1 \simeq |z_1|^{-2} L(z_1)^\intercal z_2$. For $\mathbb{R}^4$ we use $|u_1|^{-2} L(u_1)^\intercal u_2$. Then we use this multiplication to define what we call the *KS projective transformation* which allow us to separate collision and ejection orbits from near by near-collision and near by near-ejection orbits and show that, in the projectivized KS variables, *the totality of collision orbits and ejection orbits and the singularity itself* form a real analytic submanifold which we call the collision–ejection (CE) manifold. In fact in these variables, the singularity is a normally hyperbolic manifold and the CE manifold is its stable manifold. And near-collision and near-ejection orbits are repelled.

1.4. The KS Map (KSM). Before summarizing the present work, we introduce the KS map and show that it is fundamentally different from quaternionic multiplication.

Let $\mathbb{U} = \mathbb{R}^4$, $\mathbb{X} = \mathbb{R}^3 \times \{0\} \subset \mathbb{R}^4$, $\mathbb{U}^* = \mathbb{U} \setminus \{0\}$, $\mathbb{X}^* = \mathbb{X} \setminus \{0\}$ and

$$
\begin{aligned}
&L(u) := \begin{pmatrix} u_1 & -u_2 & -u_3 & u_4 \\ u_2 & u_1 & -u_4 & -u_3 \\ u_3 & u_4 & u_1 & u_2 \\ u_4 & -u_3 & u_2 & -u_1 \end{pmatrix} =: \begin{bmatrix} u^{(1)} & u^{(2)} & u^{(3)} & u^{(4)} \end{bmatrix}, \quad u \in \mathbb{U}^*, \\
&u^{(j)} := I_j u, \quad j = 1, 2, 3, 4,
\end{aligned} \tag{1.4}
$$

where the I_j's are given by (A.11). The *Kustaanheimo–Stiefel transformation* (KS) is given by

$$
\begin{aligned}
&\psi : \mathbb{U}^* \to \mathbb{X}^*, \\
&x = \psi(u) = L(u)u.
\end{aligned}
$$

The columns of $L(u)$ form a real analytic orthogonal frame for $\mathbb{U}^*$. And when $|u| = 1$, $\{u^{(2)}, u^{(3)}, u^{(4)}\}$ form a real analytic orthogonal frame for S^3. The map ψ indeed maps $\mathbb{U}^*$ to

$\mathbb{X}^*$ because the fourth component $(L(u)u)_4 = 0$. We call $\mathbb{U}^*$ the *parameter space* and $\mathbb{X}^*$ the *physical space* [14].

Now $\psi^{-1}(x)$ is the circle $\{u^{(1)}\cos t + u^{(4)}\sin t \mid t \in [-\pi, \pi)\}$ where $u \in \psi^{-1}(x)$ is arbitrary [14]. To show the difference between the KST and quaternionic multiplication we make the correspondence $u \simeq u_1 + iu_2 + ju_3 + ku_4$. If we represent quaternionic multiplication by the matrix $Q(u)$, given by (A.2), we obtain

$$\begin{aligned} &Q(u)v \simeq uv, \\ &L(u)v \simeq u\hat{v}, \quad \hat{v} = v_1 + iv_2 + jv_3 - kv_4 \simeq Nv, \\ &L(u) = Q(u)N. \end{aligned} \tag{1.5}$$

First, $\det L(u) = -|u|^4$ while $\det Q(u) = +|u|^4$. Let $\{\mathbf{e}_1, \dots, \mathbf{e}_4\}$ be the standard basis of $\mathbb{R}^4 \supset \mathbb{X}^*$ and, to avoid ambiguity, let

$$\mathbf{b}_1 = (1 \quad 0 \quad 0 \quad 0)^{\mathsf{T}}, \quad \dots, \quad \mathbf{b}_4 = (0 \quad 0 \quad 0 \quad 1)^{\mathsf{T}} \tag{1.6}$$

be the standard basis of $\mathbb{U}$. Now we compare the solutions of the equations $q(u) := Q(u)u = \pm\mathbf{e}_1$ to those of the equations $\psi(u) = \pm\mathbf{e}_1$. Since $\pm 1 \simeq \pm\mathbf{e}_1$, these are the equations that correspond to $z^2 = \pm 1$, $z \in \mathbb{C}$, for quaternionic multiplication and KS multiplication.

From (2.3) we can see that the solution to $\psi(u) = \mathbf{e}_1$ is the circle

$$\psi^{-1}(\mathbf{e}_1) = \{\mathbf{b}_1^t\} = \{u \mid u_1^2 + u_4^2 = 1,\ u_2 = u_3 = 0\}.$$

And the solution to $\psi(u) = -\mathbf{e}_1$ is also a circle

$$\psi^{-1}(-\mathbf{e}_1) = \{\mathbf{b}_2^t\} = \{u \mid u_2^2 + u_3^2 = 1,\ u_1 = u_4 = 0\}.$$

In fact, the solution to $\psi^{-1}(x)$ is always a circle. And $\mathcal{P} = (\mathbb{U}^*, \psi, \mathbb{X}^*, S^1)$ is a principal bundle.

On the other hand we can see from (A.1) that $q(u)$ does not send $\mathbb{U}^*$ to $\mathbb{X}^*$. And $q^{-1}(\mathbf{e}_1)$ is $\{\pm\mathbf{b}_1\}$, only two points. But $q^{-1}(-\mathbf{e}_1)$ is the 2-sphere

$$q^{-1}(-\mathbf{e}_1) = \{u \mid u_2^2 + u_3^2 + u_4^2 = 1,\ u_1 = 0\}.$$

Hence the dimension of $q^{-1}(u) = x$ is not constant and $q(u)$ does not define a principal bundle.

When restricted to the unit sphere S^3 the KS map becomes a Hopf map and we obtain the Hopf fibration $S^3 \to S^2$ [5,6], [12, §20], [1, p. 722].

These observations make us believe that $L(u)u$ is the appropriate generalization of the squaring map z^2 and that the KS matrices $L(u)$ are fundamentally different from the quaternion matrices $Q(u)$. "*And any attempt to substitute the theory of KS-matrices by the more popular theory of the quaternion matrices leads, therefore, to failure or at least a very unwieldy formalism*" E.L. Stiefel and G. Scheifele [14, p. 286].

Of course the KS multiplication $L(u)v$ is not associative. But let us recall that multiplication of *complex numbers* is both associative and commutative, multiplication of *quaternions* is only associative, and multiplication of *octonians* is neither.

1.5. We would like to point out that some authors make use of quaternions [15–18]. In [18] the author uses "a new elegant way of handling the three-dimensional case in complete analogy to the well-known planar case by introducing an unconventional conjugation of quaternions (see definition in Eq. (24) below), first mentioned by Waldvogel [18]". The "unconventional conjugation" is $v \mapsto v^* = v_1 + iv_2 + jv_3 - kv_4 \simeq Nv$ which is equivalent to $v \mapsto \tilde{v} = -v_1 - iv_2 - jv_3 + kv_4$, which can be found in [14, p. 286]. Then the author of [18] defines the KS map as $x = uu^*$

which, except for the notation, is nothing more than $L(u)u$, also in [14]. Then he reproduces a fragment of Chapter XI of [14] in the quaternionic notation. All this is nothing but the KS map in a cumbersome notation. It leads to the same Hopf fibration. It does not lead to the squaring of quaternions $u \mapsto u^2 \simeq Q(u)u$ because squaring of quaternions does not lead to a fibration at all as we saw above.

1.6. The LCT as well as the KST consist of a change of variables and a time rescaling. The order in which these two steps are performed does not matter. If we perform only the time rescaling we obtain

$$x' = \xi, \qquad \xi' = \frac{r'}{r}x' - \frac{a}{r}x + r^2 f$$

which is still singular [14, p. 20]. This makes the idea that only a time rescaling is needed to regularize the Kepler problem a misconception.

1.7. In Section 2 we present some known properties of the KS matrix $L(u)$ and give the fibration that the KS map induces on $\mathbb{U}^*$ [14].

1.8. In Section 3 we construct two square root branches in three dimensions. We introduce an orthonormal basis $\langle \tilde{u}_o^{(1)} \rangle = \{\tilde{u}_o^{(1)}, \tilde{u}_o^{(2)}, \tilde{u}_o^{(3)}, \tilde{u}_o^{(4)}\}$ for any fixed but arbitrary unit vector $\tilde{u}_o$. And let $x_o = L(\tilde{u}_o)(\tilde{u}_o)$. We write points as $u = \sum s_i \tilde{u}_o^{(i)} = L(\tilde{u}_o)\mathbf{s}$. Then we study what happens when we rotate the basis to $\langle \tilde{u}_o^{(1)\theta} \rangle$.

A subscript u_o indicates that we use the orthonormal basis $\langle \tilde{u}_o^{(1)} \rangle$. Define

$$\begin{aligned}
K_{u_o} &= \operatorname{span}\{\tilde{u}_o^{(1)}, \tilde{u}_o^{(2)}, \tilde{u}_o^{(3)}\} = \{v \in \mathbb{U}^* \mid v^\mathsf{T} \tilde{u}_o^{(4)} = 0\},\\
K_{u_o}^0 &= \{w \in K_{u_o} \mid w^\mathsf{T} u_o^t = 0\} = \operatorname{span}\{\tilde{u}_o^{(2)}, \tilde{u}_o^{(3)}\},\\
K_{u_o}^{\pm} &= \{w \in K_{u_o} \mid \pm w^\mathsf{T} u_o^t > 0\},\\
S_{u_o} &= \operatorname{span}\{\tilde{u}_o^{(1)}, \tilde{u}_o^{(4)}\},\\
\mathbb{U}_{u_o}^{\#} &= \mathbb{U}_{u_o}^* \setminus K_{u_o}^0,\\
\mathbb{R}_{x_o}^- &= \{x \in \mathbb{X}_{u_o}^* \mid x = a x_o,\ a \in (-\infty, 0)\},\\
\mathbb{X}_{u_o}^{\#} &= \mathbb{X}_{u_o}^* \setminus \mathbb{R}_{x_o}^- \simeq \mathbb{X}_{u_o}^{\pm},\\
\mathbb{K}_{u_o}^{\#} &= K_{u_o} \setminus K_{u_o}^0,\\
x_o &= L(u_o)u_o.
\end{aligned}$$

Recall that $0 \notin \mathbb{U}^*$ and $0 \notin \mathbb{X}^*$. Removing the ray $\mathbb{R}_{x_o}^-$ from $\mathbb{X}_{u_o}^*$ is analogous to removing a ray from $\mathbb{C}$ to obtain one of the two sheets that comprise the two-sheeted Riemann surface associated with the standard square root map.

We obtain the two real analytic square root branches $\psi_{u_o}^{\pm} : K_{u_o}^{\pm} \to \mathbb{X}_{u_o}^{\pm}$ by restricting the KSM ψ to $K_{u_o}^{\pm}$. We show that the KSM collapses every circle in the plane $K_{u_o}^0$ to a point on the ray $\mathbb{R}_{x_o}^-$.

1.9. In Section 4 we glue the two branches we defined in Section 3. We blow up $(\mathbb{R}_{x_o}^-)^{\pm}$ to $C_{u_o}^{\pm} \simeq \{0\} \times \mathbb{R}_{x_o}^- \times S^1$ and identify $(0, \gamma, a_1)^+$ with $(0, \gamma + \pi, a_1)^-$.

1.10. In Section 5 we define what we call *KS cylindrical, torical* and *spherical coordinates*. We define (ρ_1, θ, z) by

$$(s_1, s_4) = \rho_1(\cos\theta, \sin\theta), \qquad \zeta = A(-\theta)\big(s_2\tilde{u}_o^{(2)} + s_3\tilde{u}_o^{(3)}\big) = z_2\tilde{u}_o^{(2)} + z_3\tilde{u}_o^{(3)}.$$

The *KS cylindrical coordinates* are (ρ_1, θ, z).

Then for any $u \notin K^0_{u_o}$, $u_+ := \pi_{u_o}(u) := A(-\theta)u \in K^+_{u_o}$. Moreover θ is unique in $[0, 2\pi)$. It follows that $\mathcal{P}_{u_o} := (\mathbb{U}^\#_{u_o}, \pi_{u_o}, K^+_{u_o}, S^1)$ is a trivial principal bundle. That is, $\mathbb{U}^\#_{u_o} = K^+_{u_o} \times S^1$.

The *KS torical coordinates* are $(\rho_1, \theta, \rho_2, \gamma)$ where we define (ρ_2, γ) by

$$(z_2, z_3) = \rho_2(\cos\gamma, \sin\gamma).$$

The *KS spherical coordinates* are $(\rho, \theta, \gamma, \omega)$ where we define (ρ, ω) by

$$(\rho_1, \rho_2) = \rho(\cos\omega, \sin\omega).$$

We also show that $\mathcal{P} = (\mathbb{U}^*, \psi, \mathbb{X}^*, G)$ is a real analytic principal bundle.

1.11. In Section 6 we define the full square root map from an S^1-cover of $\mathbb{X}^\#$ to $\mathbb{U}^*$. We take $\mathbb{X}^\# \times S^1$ with each $\mathbb{R}^-_{x_o} \times \theta$ blown up as above. Then we make the identification

$$\begin{gathered}(r, \gamma, a; \theta) \sim \big(r', \gamma', a'; \theta'\big) \\ \Updownarrow \\ r = r' = 0, \quad a = a' < 0, \quad \gamma + \theta = \gamma' + \theta'.\end{gathered}$$

Since addition of angles is performed mod 2π, the identification in Section 4 can be written as $(\gamma, 0) \sim (\gamma + \pi, \pi)$.

2. The L Matrix

A celebrated theorem of Hurwitz states that square matrices $L(u)$ that satisfy the three properties

- $L(u)$ is orthogonal for all $u \neq \mathbf{0}$,
- $L(u)$ is linear in u, (hence $L(u') = L(u)'$ for any C^1 curve $u(t)$), and
- one of the columns of $L(u)$ is u,

can be found only in 1-, 2-, 4- and 8-dimensional spaces. Hurwitz's assertion is equivalent to saying that the only parallelizable spheres are S^0, S^1, S^3 and S^7 [2,8].

Define an antisymmetric bilinear form for $u, v \in \mathbb{U}^*$ by

$$\begin{aligned}\ell(u, v) &:= u_4v_1 - u_3v_2 + u_2v_3 - u_1v_4 = \big(L(u)v\big)_4 \\ &= v^{\mathsf{T}} I_4 u = \big(u^{(4)}, v\big).\end{aligned} \tag{2.1}$$

Corollary 2.1. [14] *Let $u \in \mathbb{U}^*$ and $x = L(u)u$ which is given by* (A.1). *Then $L(u)^{\mathsf{T}}L(u) = |u|^2 I$, $|x| = |u|^2$, $x_4 = \ell(u, u) = (L(u)u)_4 = 0$ and*

$$\begin{aligned}&\ell(u, v) = 0 \quad \textit{iff} \quad L(u)v = L(v)u, \\ &\ell(u, v) = 0 \quad \textit{iff} \quad |u|^2L(v)v - 2(u, v)L(u)v + |v|^2L(u)u = 0, \\ &\ell(u, v) = 0 \quad \textit{iff} \quad L(u)^{\mathsf{T}}L(v)v = 2(u, v)v - |v|^2u.\end{aligned} \tag{2.2}$$

Lemma 2.2. [14] *Let $y_4 = 0$ and $u \in \mathbb{U}$. Then, $\ell(u, L(u)^{\mathsf{T}} y) = 0$.*

Proof. By definition, $\ell(u, L(u)^{\mathsf{T}} y) = (L(u)L(u)^{\mathsf{T}} y)_4 = |u|^2 y_4 = 0$, since $y_4 = 0$. $\square$

Definition 2.3. Define the *KS map* by

$$\psi : \mathbb{U}^* \to \mathbb{X}^*,$$
$$\psi(u) = L(u)u.$$

Lemma 2.4 *(The fibration of $\mathbb{U}^*$).* [14] *The following are true*:

(1) *Let $x \neq y$ be two points in $\mathbb{X}^*$. Then $\psi^{-1}(x) \cap \psi^{-1}(y) = \emptyset$.*
(2) *Let $u \in \psi^{-1}(x)$ be fixed but arbitrary. Then $\psi^{-1}(x)$ is given by*

$$\psi^{-1}(x) = \{u^t = A(t)u \mid -\pi \leqslant t < \pi\},$$
$$A(t) = e^{tI_4} = I_1 \cos t + I_4 \sin t = R_{14}(-t) \oplus R_{23}(t) \tag{2.3}$$

where $A(t)$ is given explicitly in (A.9) *and the meaning of $R_{14}(-t)$ and $R_{23}(t)$ should be clear.*
(3) *$\psi^{-1}(x)$ is a circle of radius $\sqrt{r} = \sqrt{|x|} = |u|$ lying in the plane $R_u := \operatorname{span}\{u^{(1)}, u^{(4)}\}$.*
(4) *The tangent line to $\psi^{-1}(x)$ at u is given by*

$$T_u(\psi^{-1}(x)) = \operatorname{span}\{\tau_u\}, \qquad \tau_u = \left[\frac{d}{dt} A(t)u\right]_{t=0} = u^{(4)} = I_4 u.$$

(5) *It follows that*

$$\ell(u, v) = (\tau_u, v) = v^{\mathsf{T}} I_4 u, \quad u, v \in \mathbb{U}.$$

Thus v is normal to the fiber $\psi^{-1}(x)$ at $u \in \psi^{-1}(x)$ iff $\ell(u, v) = 0$.
(6) *The restriction of the KS transformation to the unit sphere is the Hopf map that takes S^3 to S^2 and the fibration is the Hopf fibration* [5,6]*,* [12, §20].

Proof. See [14]. For completeness we give a proof in our notation in Appendix A.2. $\square$

Definition 2.5. Let $G = \{A(t) \mid t \in [0, 2\pi]\}$. Define a right action of (the compact Lie group) G on $\mathbb{U}^*$ by

$$u \cdot t := u^t := A(t)u, \quad u \in \mathbb{U}^*, \ t \in [0, 2\psi]. \tag{2.4}$$

Let $[\mathbb{U}^*] = \mathbb{U}^*/G$ be the quotient space. And denote the G-orbit of $u \in \mathbb{U}^*$ by $[u]$. Sometimes we will talk about S^1 but we really mean G.

Remark. In [14, p. 271, (9)], the fiber $\psi^{-1}(x)$ is given by $\{u^{\mathsf{T}} A(t)^{-1} \mid t \in [0, 2\pi]\}$. And hence the tangent vector to the fiber at u^{T} is $-u^{\mathsf{T}} I_4 = (I_4 u)^{\mathsf{T}}$. The difference here is that we use column vectors rather than row vectors. For row vectors, right actions are defined by multiplying on the right by $A(t)^{\mathsf{T}} = A(t)^{-1} = A(-t)$ [13, p. 74]. Since u is a column vector, and since G is commutative, the action (2.4) can be viewed as a right action

$$[A(t)u]^{\mathsf{T}} = u^{\mathsf{T}} A(t)^{\mathsf{T}} = u^{\mathsf{T}} A(t)^{-1}.$$

Lemma 2.6. $\det L(u) = -|u|^4$.

Proof. Notice that $\det L(u) = \pm|u|^4$, $\mathbb{R}^4 \setminus \{0\}$ is path connected, $\det L(u)$ is continuous, and $\det L((1,0,0,0)^\mathsf{T}) = -1$. $\square$

Corollary 2.7. *The action* (2.4) *is free. That is, $A(t)u = u \Leftrightarrow A(t) = I$. And $\psi^{-1}(x) = [u]$ for any $u \in \psi^{-1}(x)$. Since G is compact and acts freely, the quotient space $[\mathbb{U}^*]$ is a real analytic manifold. $(\mathbb{U}^*, \pi, [\mathbb{U}^*], G)$ is a real analytic circle bundle. $\mathbb{U}^*$ is homeomorphic to $\mathbb{X}^*$. Hence $(\mathbb{U}^*, \psi, \mathbb{X}^*, G)$ is a C^o circle bundle.*

Lemma 2.8. *For any $u \in \mathbb{U}^*$, the equivalence class $[u] = G \cdot u$ is the unique solution of the real analytic vector field*

$$\dot{w} = I_4 w, \qquad w(0) = u. \tag{2.5}$$

The flow of this vector field is real analytic, periodic and given by

$$\mathcal{F}(u,t) = A(t)u.$$

Proof. Since $I_4^2 = -I$, we have $A(t) = e^{tI_4} = I\cos t + I_4 \sin t$. $\square$

Lemma 2.9. *Let*

$$\begin{aligned}
H_u &= \operatorname{span}\{u^{(1)}, u^{(2)}, u^{(3)}\} = \{w \in \mathbb{U}^* \mid \ell(u,w) = 0\},\\
H_u^\pm &= \{w \in H_u \mid \pm u^\mathsf{T} w > 0\},\\
H_u^0 &= \{w \in H_u \mid u^\mathsf{T} w = 0\} = \operatorname{span}\{u^{(2)}, u^{(3)}\},\\
R_u &= \operatorname{span}\{u^{(1)}, u^{(4)}\}.
\end{aligned}$$

(1) *For any $u \in \mathbb{U}$ we have $[u] = [-u] = [u^{(4)}]$ and $[u^{(2)}] = [u^{(3)}] \subset H_u^0$. In fact, $u^{(3)} = A(\pi/2)u^{(2)}$, $u^{(4)} = A(\pi/2)u^{(1)}$ and $-u^{(1)} = A(\pi)u^{(1)}$.*

(2) *It follows from* (A.12) *that*

$$\begin{aligned}
I_j H_u^0 &= R_u, \qquad I_j R_u = H_u^0, \quad j = 2,3,\\
I_4 H_u^0 &= H_u^0, \qquad I_4 R_u = R_u.
\end{aligned}$$

(3) *Thus*

$$\begin{aligned}
&(v \in H_u^0) \quad \Leftrightarrow \quad (H_u^0 = R_v) \quad \Leftrightarrow \quad (R_u = K_v^0)\\
&([v] \subset H_u^0) \quad \Leftrightarrow \quad ([u] \subset H_v^0).
\end{aligned}$$

(4) *It follows from* (A.37) *and* (A.38) *that*

$$\begin{aligned}
H_{u^t} &= A(t)H_u,\\
H_{u^t}^\pm &= A(t)H_u^\pm,\\
H_{u^t}^0 &= H_u,\\
R_{u^t} &= A(t)R_u = R_u.
\end{aligned}$$

(5) *The maps $A(t): H_u \to H_{u^t}$ and $A(t): H_u^\pm \to H_{u^t}^\pm$ are real bi-analytic section maps.*

Proof. The first, second and third assertions are obvious. The fourth is true because $u^{(4)t} = (u^t)^{(4)}$ and $A(t)^\intercal = A(t)^{-1}$. The last one is true because the vector field (2.5) is real analytic and $A(t)$ is an invertible matrix. □

Proposition 2.10. *The bundle $\mathcal{H} := \{H_u \mid u \in \mathbb{U}^*\}$ is invariant under the action of the group G. However, it is not integrable. Equivalently the vector fields $\langle u^{(1)}, u^{(2)}, u^{(3)}\rangle$ are not in involution.*

Proof. The first assertion follows from Lemma 2.9. The second can be proved in several equivalent ways. It follows from the fact that there is no C^1 function f such that $\nabla f = u^{(4)}$. Another way of looking at it is to use the identity $I_2 I_3 = I_4$ to compute the Lie bracket

$$\left[u^{(2)}, u^{(3)}\right] = I_2 u^{(3)} - I_3 u^{(2)} = (I_2 I_3 - I_3 I_2)u = 2I_4 u = u^{(4)}$$

which shows that $\mathcal{H}$ is not closed under the Lie bracket operation. □

2.11. The subbundle $\mathcal{H}$ is called the *horizontal bundle* for the KS map. It earned this name because for each $u \neq 0$, $D_u^H \psi := D_u \psi \lfloor H_u : H_u \to T_{\psi(u)}\mathbb{X}^*$ is an isomorphism.

3. Two Square Root Branches in Three Dimensions

In this section we define two square root branches by restricting the KS map to $H_{u_o}^{\pm}$ for any given but arbitrary $u_o \in \mathbb{U}^*$. We use the orthonormal basis associated with $L(\tilde{u}_o)$ where $\tilde{u}_o = |u_o|^{-1} u_o$.

3.1. Orthonormal bases associated with $\boldsymbol{u_o}$. Let u_o be fixed but arbitrary and let $x_o = L(u_o)u_o$. Define

$$\begin{aligned} &x_{u_o}^{(j)} = L(u_o)u_o^{(j)}, \\ &\tilde{u}_o^{(j)} = |u_o|^{-1} u_o^{(j)}, \\ &\tilde{x}_{u_o}^{(j)} = |x|^{-1} x_{u_o}^{(j)}, \quad j = 1, 2, 3, 4. \end{aligned} \tag{3.1}$$

Let $\mathbb{U}_{u_o}^*$ be $\mathbb{U}^*$ equipped with the orthonormal basis $\langle \tilde{u}_o\rangle_4 := \{\tilde{u}_o^{(1)}, \tilde{u}_o^{(2)}, \tilde{u}_o^{(3)}, \tilde{u}_o^{(4)}\}$. And let K_{u_o} be H_{u_o} equipped with the orthonormal basis $\langle \tilde{u}_o\rangle_3 := \{\tilde{u}_o^{(1)}, \tilde{u}_o^{(2)}, \tilde{u}_o^{(3)}\}$. Similarly, define $K_{u_o}^q$ from $H_{u_o}^q$, $q = 0, \pm$. Finally, let S_{u_o} be R_{u_o} equipped with the orthonormal basis $\{\tilde{u}_o^{(1)}, \tilde{u}_o^{(4)}\}$.

3.2. The $\boldsymbol{u_o}$-orthogonal direct sum decomposition. It follows that $\mathbb{U}_{u_o}^*$ is the direct sum of two orthogonal subspaces:

$$\begin{aligned} &\mathbb{U}_{u_o}^* \simeq S_{u_o} \oplus K_{u_o}^0, \\ &u = \eta + \xi, \quad \eta \in S_{u_o}, \ \xi \in K_{u_o}^0. \end{aligned}$$

We write $u \in \mathbb{U}^*$ in several ways:

$$\begin{aligned} u &= s_1 \tilde{u}_o^{(1)} + s_2 \tilde{u}_o^{(2)} + s_3 \tilde{u}_o^{(3)} + s_4 \tilde{u}_o^{(4)} \\ &= L(\tilde{u}_o)\mathbf{s}_{u_o} \\ &= \eta + \xi, \\ \eta &= s_1 \tilde{u}_o^{(1)} + s_4 \tilde{u}_o^{(4)}, \ \xi = s_2 \tilde{u}_o^{(2)} + s_3 \tilde{u}_o^{(3)}, \ \mathbf{s}_{u_o} = L(\tilde{u}_o)^\intercal u. \end{aligned} \tag{3.2}$$

We identify the orthonormal basis $\langle \tilde{u}_o \rangle_4$ with the orthonormal matrix $L(\tilde{u}_o)$. We identify the orthonormal basis $\{\tilde{u}_o^{(1)}, \tilde{u}_o^{(4)}\}$ with

$$L_{14}(\tilde{u}_o) := \begin{bmatrix} \tilde{u}_o^{(1)} & \tilde{u}_o^{(4)} \, t \end{bmatrix} \simeq \begin{bmatrix} \tilde{u}_o^{(1)} & 0 & 0 & \tilde{u}_o^{(4)} \end{bmatrix} \tag{3.3}$$

and the orthonormal basis $\{\tilde{u}_o^{(2)}, \tilde{u}_o^{(3)}\}$ with

$$L_{23}(\tilde{u}_o) := \begin{bmatrix} \tilde{u}_o^{(2)} & \tilde{u}_o^{(3)} \end{bmatrix} \simeq \begin{bmatrix} 0 & \tilde{u}_o^{(2)} & \tilde{u}_o^{(3)} & 0 \end{bmatrix}. \tag{3.4}$$

We write

$$\begin{aligned} s_1 \tilde{u}_o^{(1)} + s_4 \tilde{u}_o^{(4)} &= \begin{bmatrix} \tilde{u}_o^{(1)} & \tilde{u}_o^{(4)} \end{bmatrix} \begin{pmatrix} s_1 \\ s_4 \end{pmatrix} = L_{14}(\tilde{u}_o) \begin{pmatrix} s_1 \\ s_4 \end{pmatrix} \\ &= \begin{bmatrix} \tilde{u}_o^{(1)} & 0 & 0 & \tilde{u}_o^{(4)} \end{bmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix} = L_{14}(\tilde{u}_o) \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix} \end{aligned}$$

where by abuse of notation we use $L_{14}(\tilde{u}_o)$ to denote the two involved matrices. Similar convention is used for $L_{23}(\tilde{u}_o)$.

3.3. The $\boldsymbol{u_o}$-orthonormal basis for $\mathbb{X}^*$. Let $\mathbb{X}^*_{u_o}$ be $\mathbb{X}^*$ equipped with the orthonormal basis $\langle \tilde{x}_{u_o} \rangle := \{\tilde{x}_{u_o}^{(1)}, \tilde{x}_{u_o}^{(2)}, \tilde{x}_{u_o}^{(3)}\}$. Recall that $\mathbb{X}^* = \mathbb{R}^3 \times \{0\} \subset \mathbb{R}^4$. Let

$$\begin{aligned} Y(\tilde{u}_o) &= \begin{bmatrix} \tilde{x}_{u_o}^{(1)} & \tilde{x}_{u_o}^{(2)} & \tilde{x}_{u_o}^{(3)} & \tilde{x}_{u_o}^{(4)} \end{bmatrix}, \\ X(\tilde{u}_o) &= \begin{bmatrix} \tilde{x}_{u_o}^{(1)} & \tilde{x}_{u_o}^{(2)} & \tilde{x}_{u_o}^{(3)} & 0 \end{bmatrix}, \\ C(\tilde{u}_o) &= \begin{bmatrix} \tilde{x}_{u_o}^{(1)} & \tilde{x}_{u_o}^{(2)} & \tilde{x}_{u_o}^{(3)} \end{bmatrix}. \end{aligned} \tag{3.5}$$

It follows that

$$L(\tilde{u}_o)^2 = Y(\tilde{u}_o). \tag{3.6}$$

We identify the orthonormal basis $\langle \tilde{x}_{u_o}^{(1)}, \tilde{x}_{u_o}^{(2)}, \tilde{x}_{u_o}^{(3)}, \tilde{x}_{u_o}^{(4)} \rangle$ with the orthonormal matrix $Y(\tilde{u}_o)$. We identify the orthonormal basis $\langle \tilde{x}_{u_o}^{(1)}, \tilde{x}_{u_o}^{(2)}, \tilde{x}_{u_o}^{(3)} \rangle$ with either of the two orthonormal matrices $X(\tilde{u}_o)$ or $C(\tilde{u}_o)$.

3.4. Cayley matrices. The fourth row of $C(\tilde{u}_o)$ is a zero row. If we remove it, since $\tilde{u}_o$ is a unit vector, the 3×3 orthonormal matrix that remains is called a Cayley matrix. This is a traditional way of representing points on S^3 by 3×3 orthonormal matrices. By abuse of notation we will continue to denote both the 3×3 and 4×3 matrices by $C(b)$ where $b \in S^3$.

3.5. While we are at it we find what happens to the expression $u = L(\tilde{u}_o)\mathbf{s}_{u_o}$ when we rotate u_o, u, or both. First notice that it follows from Lemma 2.9 that

$$S_{u_o^\beta} = A(\beta) S_{u_o} = S_{u_o}, \qquad K^0_{u_o^\beta} = A(\beta) K_{u_o} = K_{u_o}, \qquad \mathbb{U}^*_{u_o^\beta} = \mathbb{U}^*_{u_o}. \tag{3.7}$$

Thus,

$$\mathbb{U}^*_{u_o^\beta} \simeq S_{u_o^\beta} \oplus K^0_{u_o^\beta}$$

and $u \in \mathbb{U}^*_{u_o}$ has a unique decomposition

$$u = \eta_{u_o^\beta} + \xi_{u_o^\beta}, \qquad \eta_{u_o^\beta} = \eta \in S_{u_o^\beta}, \qquad \xi_{u_o^\beta} = \xi \in K^0_{u_o^\beta}.$$

The basis is what makes $K^0_{u_o}$ different from $K^0_{u_o^\beta}$ and S_{u_o} from $S_{u_o^\beta}$.

3.6. The components of u^β in the orthonormal basis $\langle \tilde{u}_o \rangle_4$. Write $u^\beta = L(\tilde{u}_o)\mathbf{s}^\beta_{u_o}$. We use (A.34) to compute $\mathbf{s}^\beta_{u_o}$:

$$\begin{aligned} u^\beta &= L(\tilde{u}_o)B(\beta)\mathbf{s}_{u_o} = L(\tilde{u}_o)\mathbf{s}^\beta_{u_o}, \\ \mathbf{s}^\beta_{u_o} &= B(\beta)\mathbf{s}_{u_o}. \end{aligned} \tag{3.8}$$

3.7. The components of u in the rotated frame $\langle \tilde{u}^\beta_o \rangle_4$. It follows from (A.38) that the orthonormal basis $\langle \tilde{u}^\beta_o \rangle_4$ is given by

$$\langle u^\beta_o \rangle_4 = \langle u_o \rangle_4 A(-\beta) \quad \Longleftrightarrow \quad L(\tilde{u}^\beta_o) = L(\tilde{u}_o)A(-\beta). \tag{3.9}$$

Thus $\mathbf{s}_{u^\beta_o}$, which represents u in the orthonormal basis $\langle \tilde{u}^\beta_o \rangle_4$, is computed as follows:

$$\begin{aligned} u &= L(\tilde{u}_o)\mathbf{s}_{u_o} \\ &= L(\tilde{u}^\beta_o)A(\beta)\mathbf{s}_{u_o}, \\ \mathbf{s}_{u^\beta_o} &= A(\beta)\mathbf{s}_{u_o}, \\ \begin{pmatrix} s_1 \\ s_4 \end{pmatrix}_\beta &= R(-\beta)\begin{pmatrix} s_1 \\ s_4 \end{pmatrix}, \\ \begin{pmatrix} s_2 \\ s_3 \end{pmatrix}_\beta &= R(\beta)\begin{pmatrix} s_2 \\ s_3 \end{pmatrix}. \end{aligned} \tag{3.10}$$

3.8. Rotating both the frame $\langle \tilde{u}_o \rangle_4$ and u by the same angle β. Finally we use (A.38) and (A.34) to compute the components of u^β in the orthonormal basis $\langle \tilde{u}^\beta_o \rangle_4$:

$$\begin{aligned} u^\beta &= A(\beta)u = A(\beta)L(\tilde{u}_o)\mathbf{s}_{u_o} \\ &= L(\tilde{u}_o)B(\beta)\mathbf{s}_{u_o} \\ &= L(\tilde{u}^\beta_o)A(\beta)B(\beta)\mathbf{s}_{u_o}, \\ A(\beta)B(\beta) &= \big[R_{14}(-\beta) \oplus R_{23}(\beta)\big]\big[R_{14}(\beta) \oplus R_{23}(\beta)\big] \\ &= R_{14}(0) \oplus R_{23}(2\beta), \\ \mathbf{s}^\beta_{u^\beta_o} &= C(\beta)\mathbf{s}_{u_o}, \\ C(\beta) &= R_{14}(0) \oplus R_{23}(2\beta) \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\beta & -\sin 2\beta & 0 \\ 0 & \sin 2\beta & \cos 2\beta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned} \tag{3.11}$$

3.9. The KS map in the orthonormal basis $\langle \tilde{u}^{(1)}_o, \tilde{u}^{(2)}_o, \tilde{u}^{(3)}_o, \tilde{u}^{(4)}_o \rangle$. Let $u = L(\tilde{u}_o)\mathbf{s}$ and $v = L(\tilde{u}_o)\mathbf{r}$. Then

$$\begin{aligned} L(u)v &= L\big(L(\tilde{u}_o)\mathbf{s}\big)L(\tilde{u}_o)\mathbf{r} \\ &= L(\tilde{u}_o)NL(N\mathbf{s})L(\tilde{u}_o)\mathbf{r} \quad \text{by (A.19)} \\ &= L(\tilde{u}_o)L(\tilde{u}_o)L(N\mathbf{s})N\mathbf{r} \quad \text{by (A.18)} \\ &= Y(\tilde{u}_o)L(N\mathbf{s})N\mathbf{r} \quad \text{by (3.6)}. \end{aligned} \tag{3.12}$$

And the KS map takes the form

$$L(u)u = Y(\tilde{u}_o)L(N\mathbf{s})N\mathbf{s}.$$

Since $(L(b)b)_4 = 0$, we have

$$\begin{aligned} L(u)u &= \begin{bmatrix} \tilde{x}^{(1)}_{u_o} & \tilde{x}^{(2)}_{u_o} & \tilde{x}^{(3)}_{u_o} & 0 \end{bmatrix} L(N\mathbf{s})N\mathbf{s} \\ &= X(\tilde{u}_o)L(N\mathbf{s})N\mathbf{s}, \end{aligned}$$

$$L(N\mathbf{s})N\mathbf{s} = \begin{pmatrix} s_1^2 - s_2^2 - s_3^2 + s_4^2 \\ 2(s_1s_2 + s_3u_4) \\ 2(s_1s_3 - s_2s_4) \\ 0 \end{pmatrix} = \begin{pmatrix} |\eta|^2 - |\xi|^2 \\ 2\eta \cdot \xi \\ 2\eta \times \xi \\ 0 \end{pmatrix}. \tag{3.13}$$

In the last step we make the identifications $\eta = s_1\tilde{u}^{(1)}_o + s_4\tilde{u}^{(4)}_o \simeq (s_1, s_4), \ldots.$

Expressing $L(u)v$ in the form (3.12) is useful when we consider the restriction of the derivative of the KS map to the horizontal bundle:

$$\begin{aligned} &D^H\psi : \mathcal{H} \to T\mathbb{R}^4, \\ &D^H\psi(u, v) = 2L(u)v. \end{aligned}$$

3.10. Define the open sets

$$\mathbb{U}^{\#}_{u_o} = \mathbb{U}^*_{u_o} \setminus K^0_{u_o}, \qquad \mathbb{K}^{\#}_{u_o} = K_{u_o} \setminus K^0_{u_o}.$$

3.11. A cut in $\mathbb{X}^*$. We begin the construction of the basic square root branch by removing a ray from $\mathbb{X}^*_{u_o}$ in analogy to what we do to $\mathbb{C}$ to define a square root branch for the complex square root. Recall that $0 \notin \mathbb{U}^*$ and $0 \notin \mathbb{X}^*$. Let

$$\begin{aligned} &\mathbb{R}^-_{x_o} = \left\{ x \in \mathbb{X}^*_{u_o} \;\middle|\; x = a x_o,\ a \in (-\infty, 0) \right\}, \\ &\mathbb{X}^{\#}_{u_o} = \mathbb{X}^*_{u_o} \setminus \mathbb{R}^-_{x_o}. \end{aligned} \tag{3.14}$$

Corollary 3.12. *It follows from* (3.13) *that the KS map maps $\mathbb{U}^{\#}_{u_o}$ onto $\mathbb{X}^{\#}_{u_o}$, $\mathbb{K}^{\#}_{u_o}$ onto $\mathbb{X}^{\#}_{u_o}$, and $K^0_{u_o}$ onto $\mathbb{R}^-_{x_o}$.*

3.13. The restriction of the KS map to K_{u_o}. In K_{u_o}, $u_4 = 0$. We write points in K_{u_o} as

$$u = s_1\tilde{u}^{(1)}_o + z_2\tilde{u}^{(2)}_o + z_3\tilde{u}^{(3)}_o = s_1\tilde{u}^{(1)}_o + \zeta.$$

Then (3.13) takes the form

$$\begin{aligned} &\hat{\psi}_{u_o} := \psi \lfloor K_{u_o} \to \mathbb{X}^*_{u_o}, \\ &L(u)u = X(\tilde{u}_o)L(N\mathbf{s})N\mathbf{s} = X(\tilde{u}_o)\mathbf{a}, \end{aligned}$$

$$\mathbf{a} = L(N\mathbf{s})N\mathbf{s} = \begin{pmatrix} s_1^2 - z_2^2 - z_3^2 \\ 2s_1z_2 \\ 2s_1z_3 \\ 0 \end{pmatrix} = \begin{pmatrix} |s_1|^2 - |\zeta|^2 \\ 2s_1\xi \\ 0 \end{pmatrix} \tag{3.15}$$

where we combine the second and third components together. Notice that $\mathbf{a} = (-|z|^2, 0, 0, 0)$ when $u \in K^0_{u_o}$.

Given $x = X(\tilde{u}_o)\mathbf{a} \in \mathbb{X}^*_{u_o}$, if we try to find $(\hat{\psi}_{u_o})^{-1}(x)$, we obtain two solutions when $x \notin \mathbb{R}^-_{x_o}$ and a circle when $x \in \mathbb{R}^-_{x_o}$. More precisely,

$$\begin{aligned}
&\chi^{\pm}_{u_o}(x) = s_1\tilde{u}^{(1)}_o + z_2\tilde{u}^{(2)}_o + z_3\tilde{u}^{(3)}_o, \quad x \notin \mathbb{R}^-_{x_o},\\
&s_1 = \pm\sqrt{\frac{a_1 + |\mathbf{a}|}{2}} = \pm\rho_1,\\
&z_2 = \frac{a_2}{2s_1} = \pm\frac{a_2}{2\rho_1},\\
&z_3 = \frac{a_3}{2s_1} = \pm\frac{a_3}{2\rho_1},\\
&S(x) = \left\{\sqrt{|a_1|}\tilde{u}^{(2)\beta}_o \;\middle|\; \beta \in [-\pi, \pi)\right\}, \quad x \in \mathbb{R}^-_{x_o}.
\end{aligned} \tag{3.16}$$

If we write

$$\rho_2 = |z|, \qquad \alpha = \sqrt{a_2^2 + a_3^2}$$

then

$$\begin{aligned}
&\rho_2 = \frac{\alpha}{2\rho_1} = \frac{\alpha}{\sqrt{2(a_1 + |\mathbf{a}|)}},\\
&a_1 + i\alpha = (\rho_1 + i\rho_2)^2,\\
&\rho_1 + i\rho_2 = \sqrt{a_1 + i\alpha}.
\end{aligned} \tag{3.17}$$

The last square root is well defined because $\rho_2 > 0$.

3.14. Two square root branches associated with u_o. We define two copies of $\mathbb{X}^{\#}_{u_o}$ by

$$\mathbb{X}^{\pm}_{u_o} = \mathbb{X}^{\#}_{u_o} \times \{\pm\}. \tag{3.18}$$

We will denote $(x, \pm)$ by $x^{\pm}$. When it does not lead to ambiguity we drop the superscript "±" and write $x \in \mathbb{X}^{\pm}_{u_o} = \mathbb{X}^*_{u_o} \setminus \mathbb{R}^-_{x_o}$. We equip $\mathbb{X}^{\pm}_{u_o}$ with the orthonormal basis $\{\tilde{x}^{(1)}_{u_o}, \tilde{x}^{(2)}_{u_o}, \tilde{x}^{(3)}_{u_o}\}^{\pm}$.

Since $K^+_{u_o}$ and $K^-_{u_o}$ are disjoint, we can obtain two separate real bi-analytic maps $\psi^+_{u_o}$ and $\psi^-_{u_o}$, out of the map $\hat{\psi}_{u_o}$. Their inverses, $\chi^+_{u_o}$ and $\chi^-_{u_o} = -\chi^+_{u_o}$, give two 3-dimensional square root branches:

$$\begin{aligned}
&\psi^{\pm}_{u_o} = \hat{\psi}_{u_o}\lfloor K^{\pm}_{u_o} : K^{\pm}_{u_o} \to \mathbb{X}^{\pm}_{u_o}\rfloor,\\
&\chi^{\pm}_{u_o} = \left(\psi^{\pm}_{u_o}\right)^{-1} : \mathbb{X}^{\pm}_{u_o} \to K^{\pm}_{u_o},\\
&\chi^-_{u_o} = -\chi^+_{u_o}.
\end{aligned} \tag{3.19}$$

Together $\psi^+_{u_o}$ and $\psi^-_{u_o}$ give us a real bi-analytic map from the disjoint union $K^+_{u_o} \sqcup K^-_{u_o}$ onto the disjoint union $\mathbb{X}^+_{u_o} \sqcup \mathbb{X}^-_{u_o}$. By suggestive abuse of notation we denote the combined map by $\psi^{\pm}_{u_o}$ and its inverse by $\chi^{\pm}_{u_o}$:

$$\begin{aligned}
&\psi^{\pm}_{u_o} : K^+_{u_o} \sqcup K^-_{u_o} \to \mathbb{X}^+_{u_o} \sqcup \mathbb{X}^-_{u_o},\\
&\chi^{\pm}_{u_o} : \mathbb{X}^+_{u_o} \sqcup \mathbb{X}^-_{u_o} \to K^+_{u_o} \sqcup K^-_{u_o}.
\end{aligned} \tag{3.20}$$

4. Gluing the two square root branches

In this section we glue the two maps $\chi_{u_o}^{+}$ and $\chi_{u_o}^{-}$ together to obtain a square root map on $\mathbb{R}^3$ in a fashion similar to gluing the two branches $\pm\sqrt{z}$, $z \in \mathbb{C}$, together to obtain one map defined on a two-sheeted Riemann surface.

4.1. Standard spherical coordinates on K_{u_o} and $\mathbb{X}_{u_o}^{\pm}$. To simplify calculations we introduce standard spherical coordinates on K_{u_o} and $\mathbb{X}_{u_o}^{\pm}$ as follows:

(1) On K_{u_o} we make the identification $u \simeq (z_2, z_3, s_1)$ and use (ρ, γ, κ), where $\tilde{u}_o^{(1)}$ is the vertical (traditional z-)axis; $\gamma \in [-\pi, \pi)$; $\kappa \in [0, \pi]$; $\rho = \sqrt{\rho_1^2 + \rho_2^2} \geqslant 0$; and

$$z = \rho_2(\cos\gamma, \sin\gamma), \qquad (s_1, \rho_2) = \rho(\cos\kappa, \sin\kappa). \tag{4.1}$$

We also use standard cylindrical coordinates (ρ_2, γ, s_1). We write $u \in K_{u_o}$ in the form

$$u = s_1\tilde{u}_o^{(1)} + \rho_2\tilde{u}_o^{(2)\gamma}, \quad u \in K_{u_o}.$$

(2) On $\mathbb{X}_{u_o}^{\pm}$ we use (r, μ, ν), where $\tilde{x}_{u_o}^{(1)}$ is the vertical (traditional z-)axis; $\mu \in [-\pi, \pi)$; $\nu \in [0, \pi]$; $r = |\mathbf{a}|$; and

$$(a_2, a_3) = \alpha(\cos\mu, \sin\mu), \qquad (a_1, \alpha) = \rho(\cos\nu, \sin\nu). \tag{4.2}$$

We also use standard cylindrical coordinates (α, μ, a_1).

(3) We will be working with several coordinate systems. When variables take specific values and ambiguity might strike, we write $(\cdot, \cdot, \cdot)_c$ for cylindrical coordinates; $(\cdot, \cdot, \cdot)_s$ for spherical; $(\cdot, \cdot, \cdot)_r$ for rectangular.

4.2. Since we write $u \in K_{u_o}$ in the form $u = s_1\tilde{u}_o^{(1)} + \rho_2\tilde{u}_o^{(2)\gamma}$, we need to express $x = \psi(u)$ in a similar form. Thus we need to know the effect of γ on $\tilde{x}_{u_o}^{(2)}$ and $\tilde{x}_{u_o}^{(3)}$. Although we compute $\tilde{u}_o^{(2)\cdot\gamma}$ and $\tilde{u}_o^{(3)\cdot\gamma}$ in (3.9), we define $\tilde{x}_{u_o}^{(2)\cdot\gamma}$ and $\tilde{x}_{u_o}^{(3)\cdot\gamma}$ by

$$\begin{aligned} \tilde{x}_{u_o}^{(2)\cdot\gamma} &= L(\tilde{u}_o)\tilde{u}_o^{(2)\cdot\gamma}, \\ \tilde{x}_{u_o}^{(3)\cdot\gamma} &= L(\tilde{u}_o)\tilde{u}_o^{(3)\cdot\gamma}. \end{aligned} \tag{4.3}$$

That is

$$\begin{pmatrix} \tilde{x}_{u_o}^{(2)\cdot\gamma} & \tilde{x}_{u_o}^{(3)\cdot\gamma} \end{pmatrix} = \begin{pmatrix} \tilde{x}_{u_o}^{(2)} & \tilde{x}_{u_o}^{(3)} \end{pmatrix} R(\gamma). \tag{4.4}$$

It follows that any $x \in \mathbb{X}_{u_o}^{*}$ can be written as

$$\begin{aligned} x &= a_1\tilde{x}_{u_o}^{(1)} + \alpha\tilde{x}_{u_o}^{(2)\cdot\mu}, \quad \alpha \neq 0 \\ &= r\big(\tilde{x}_{u_o}^{(1)}\cos\nu + \tilde{x}_{u_o}^{(2)\cdot\mu}\sin\nu\big) \\ &\simeq (\alpha, \mu, a_1)_c \\ &\simeq (r, \mu, \nu)_s \\ x &= a_1\tilde{x}_{u_o}^{(1)}, \quad \alpha = 0. \end{aligned} \tag{4.5}$$

Thus we will always write

$$x = a_1\tilde{x}_{u_o}^{(1)} + \alpha\tilde{x}_{u_o}^{(2)\cdot\mu}$$

with the understanding that the second term drops out when $\alpha = 0$.

4.3. In the standard spherical and cylindrical coordinates of Section 4.1 $\hat{\psi}_{u_o}$ takes the form

$$
\begin{aligned}
&\hat{\psi}_{u_o} := \psi\lfloor K_{u_o} \to \mathbb{X}^*_{u_o},\\
&\hat{\psi}_{u_o}(u) = \rho^2\big(\tilde{x}^{(1)}_{u_o}\cos 2\kappa + \tilde{x}^{(2)\cdot\gamma}_{u_o}\sin 2\kappa\big)\\
&\qquad = r\big(\tilde{x}^{(1)}_{u_o}\cos\nu + \tilde{x}^{(2)\cdot\mu}_{u_o}\sin\nu\big)\\
&\qquad = \big(s_1^2-\rho_2^2\big)\tilde{x}^{(1)}_{u_o} + 2s_1\rho_2\tilde{x}^{(2)\cdot\gamma}_{u_o}\\
&\qquad = \big(\rho_1^2-\rho_2^2\big)\tilde{x}^{(1)}_{u_o} + 2\rho_1\rho_2\tilde{x}^{(2)\cdot\mu}_{u_o}\\
&\qquad = a_1\tilde{x}^{(1)}_{u_o} + \alpha\tilde{x}^{(2)\cdot\mu}_{u_o},\\
&(r,\mu,\nu) = \begin{cases} (\rho^2,\gamma,2\kappa), & 0\leqslant\kappa<\pi/2,\\ (\rho^2,\gamma+\pi,2(\pi-\kappa)), & \pi/2<\kappa\leqslant\pi,\\ -\rho^2\tilde{x}^{(1)}_{u_o}\simeq(\rho^2,?,\pi), & \kappa=\pi/2,\end{cases}\\
&(\alpha,\mu,s_1) = \begin{cases} (2\rho_1\rho_2,\gamma,\rho_1^2-\rho_2^2), & s_1>0,\\ (2\rho_1\rho_2,\gamma+\pi,\rho_1^2-\rho_2^2), & s_1<0,\\ -\rho^2\tilde{x}^{(1)}_{u_o}\simeq(0,?,-\rho_2^2), & s_1=0,\end{cases}
\end{aligned}
\tag{4.6}
$$

where "?" indicates that μ is not well defined when $\kappa=\pi/2$.

Notice that when $\pi/2<\kappa\leqslant\pi$, $\sin 2\kappa<0$ and $s_1<0$. Also recall that $A(\pi)=-I$. Thus $A(\mu)\sin\nu = A(\gamma)\sin 2\kappa$ in both $K^+_{u_o}$ and $K^-_{u_o}$. In $K^0_{u_o}$, $\kappa=\pi/2$ and $\sin 2\kappa=0$ and $\cos 2\kappa=-1$. Thus the first and second forms of $\hat{\psi}_{u_o}$ are valid in all of K_{u_o}.

Moreover, since if $u\simeq(\rho,\gamma,\kappa)$, $-u\simeq(\rho,\gamma+\pi,\pi-\kappa)$, we have

$$\hat{\psi}_{u_o}(u) = \hat{\psi}_{u_o}(-u).$$

Notice that $\hat{\psi}_{u_o}$ maps $K^0_{u_o}$ onto the ray $\mathbb{R}^-_{x_o}$.

It is obvious now that $\hat{\psi}_{u_o}:\mathbb{K}^\#_{u_o}\to\mathbb{X}^\#_{u_o}$ is a double cover and that

$$\psi^m_{u_o} := \psi\lfloor P^\gamma_{u_o}\to Q^\gamma_{u_o}$$

is a double cover of the form $z\mapsto z^2$, $z\in\mathbb{C}$, where $P^\gamma_{u_o}$ and $Q^\gamma_{u_o}$ are given in article Section 5.14.

4.4. From (4.6) $\psi^\pm_{u_o}$ takes the form

$$
\begin{aligned}
&\psi^\pm_{u_o}(u) = \big(s_1^2-\rho_2^2\big)\tilde{x}^{(1)}_{u_o} + 2s_1\rho_2\tilde{x}^{(2)\cdot\gamma}_{u_o}\\
&\qquad = \big(\rho_1^2-\rho_2^2\big)\tilde{x}^{(1)}_{u_o} + 2\rho_1\rho_2\tilde{x}^{(2)\cdot\mu}_{u_o}\\
&\qquad = a_1\tilde{x}^{(1)}_{u_o} + \alpha\tilde{x}^{(2)\cdot\mu}_{u_o},\\
&(\alpha,\mu,a_1)^\pm = \begin{cases} (2\rho_1\rho_2,\gamma,\rho_1^2-\rho_2^2)^+, & s_1=\rho_1>0,\\ (2\rho_1\rho_2,\gamma+\pi,\rho_1^2-\rho_2^2)^-, & s_1=-\rho_1<0.\end{cases}
\end{aligned}
\tag{4.7}
$$

As for the inverse, we have

$$
\begin{aligned}
&\chi^\pm_{u_o}\big(x^\pm\big) = u^\pm, \quad x\notin\mathbb{R}^-_{x_o}\\
&\qquad = \pm\rho_1\tilde{u}^{(1)}_o + \rho_2\tilde{u}^{(2)\gamma}_o,\\
&x^\pm = r\big(\tilde{x}^{(1)}_{u_o}\cos\nu + \tilde{x}^{(2)\cdot\mu}_{u_o}\sin\nu\big)^\pm,
\end{aligned}
$$

$$
\begin{aligned}
u^+ &= \sqrt{r}\big(\tilde{u}_o^{(1)}\cos(\nu/2) + \tilde{u}_o^{(2)\mu}\sin(\nu/2)\big) \\
&= \rho_1\tilde{u}_o^{(1)} + \rho_2\tilde{u}_o^{(2)\gamma} \\
&= \rho_1\tilde{u}_o^{(1)} + \rho_2\tilde{u}_o^{(2)\mu}, \\
u^- &= \sqrt{r}\big(\tilde{u}_o^{(1)}\cos(\pi-\nu/2) + \tilde{u}_o^{(2)(\mu+\pi)}\sin(\pi-\nu/2)\big) \\
&= -\rho_1\tilde{u}_o^{(1)} + \rho_2\tilde{u}_o^{(2)(\mu+\pi)} \\
&= -\rho_1\tilde{u}_o^{(1)} - \rho_2\tilde{u}_o^{(2)\gamma} \\
&= -u^+, \\
(\rho, s_1, \gamma, \kappa) &= \begin{cases} (\sqrt{r}, \rho_1, \mu, \nu/2), & x = x^+ \in \mathbb{X}_{u_o}^+, \\ (\sqrt{r}, -\rho_1, \mu+\pi, \pi-\nu/2), & x = x^- \in \mathbb{X}_{u_o}^-, \end{cases} \\
a_1 + i\alpha &= (\rho_1 + i\rho_2)^2, \\
\rho_1 + i\rho_2 &= \sqrt{a_1 + i\alpha}.
\end{aligned} \tag{4.8}
$$

As usual, addition of angles is performed mod 2π.

4.5. We would like to extend $\psi_{u_o}^{\pm}$ to $\widetilde{K}_{u_o}^{\pm} := K_{u_o}^{\pm} \sqcup K_{u_o}^0$.

(1) From (4.7) we obtain the following two limits as $u \in K_{u_o}^{\pm}$ approaches $K_{u_o}^0$ from above and below:

$$
\begin{aligned}
\psi_{u_o}^+(u) &= \big(2\rho_1\rho_2, \gamma, \rho_1^2 - \rho_2^2\big)_c^+ \to \big(0, \gamma, -\rho_2^2\big)_c^+ \quad \text{as } s_1 \searrow 0, \\
\psi_{u_o}^-(u) &= \big(2\rho_1\rho_2, \gamma+\pi, \rho_1^2 - \rho_2^2\big)_c^- \to \big(0, \gamma+\pi, -\rho_2^2\big)_c^- \quad \text{as } s_1 \nearrow 0.
\end{aligned} \tag{4.9}
$$

(2) Let

$$
\begin{aligned}
\widetilde{\mathbb{X}}_{u_o}^{\pm} &\simeq \mathbb{X}_{u_o}^{\pm} \sqcup C_{u_o}^{\pm}, \\
C_{u_o}^{\pm} &= \big\{(\alpha, \mu, a_1)^{\pm} \;\big|\; \alpha = 0,\ \mu \in [-\pi, \pi),\ a_1 < 0\big\} \\
&= \big\{(r, \mu, \nu)^{\pm} \;\big|\; r > 0,\ \mu \in [-\pi, \pi),\ \nu = \pi\big\}.
\end{aligned} \tag{4.10}
$$

(3) Now we can extend $\psi_{u_o}^{\pm}$ to $\widetilde{K}_{u_o}^{\pm}$ as follows:

$$
\begin{aligned}
&\tilde{\psi}_{u_o}^{\pm} : \widetilde{K}_{u_o}^{\pm} := K_{u_o}^{\pm} \sqcup K_{u_o}^0 \to \widetilde{\mathbb{X}}_{u_o}^{\pm}, \\
&\tilde{\psi}_{u_o}^{\pm}(u) = \begin{cases} \psi_{u_o}^{\pm}(u), & s_1 \neq 0, \\ (0, \mu^{\pm}, a_1)^{\pm}, & s_1 \neq 0, \end{cases} \\
&\mu = \mu^{\pm} = \begin{cases} \gamma, & u \in \widetilde{K}_{u_o}^+, \\ \gamma + \pi, & u \in \widetilde{K}_{u_o}^-. \end{cases}
\end{aligned} \tag{4.11}
$$

We emphasize here that the angles γ and μ are defined relative to the u_o-bases that we equip $\widetilde{K}_{u_o}^{\pm}$ and $\widetilde{\mathbb{X}}_{u_o}^{\pm}$ with.

(4) To extend their inverses $\chi_{u_o}^{\pm}$, given by (4.8), to $\widetilde{\mathbb{X}}_{u_o}^{\pm}$, first we note that $A(\pi) = -I$ and

$$
(\alpha \to 0, a_1 > 0) \quad \Leftrightarrow \quad \rho_2 \searrow 0, \qquad (\alpha \to 0, a_1 < 0) \quad \Leftrightarrow \quad |s_1| = \rho_1 \searrow 0.
$$

Now define

$$\begin{aligned}
&\tilde{\chi}^{\pm}_{u_o} : \widetilde{\mathbb{X}}^{\pm}_{u_o} \to \widetilde{K}^{\pm}_{u_o},\\
&\tilde{\chi}^{\pm}_{u_o}(x) = \pm\rho_1 \tilde{u}^{(1)}_o + \rho_2 \tilde{u}^{(2)\gamma}_o\\
&\qquad = \begin{cases} \chi^{\pm}_{u_o}(x^{\pm}), & x^{\pm} \in \mathbb{X}^{\pm}_{u_o},\\ (\sqrt{|a_1|}, \gamma^{\pm}, 0)_c, & x^{\pm} = (0, \mu, a_1)^{\pm} \in C^{\pm}_{u_o},\end{cases}\\
&\gamma = \gamma^{\pm} = \begin{cases} \mu, & x^{+} \in \widetilde{\mathbb{X}}^{+}_{u_o},\\ \mu + \pi, & x^{-} \in \widetilde{\mathbb{X}}^{-}_{u_o}.\end{cases}
\end{aligned} \tag{4.12}$$

(5) It follows from (4.12) that

$$\tilde{\chi}^{+}_{u_o}\big((0, \mu, a_1)^{+}_{c}\big) = \tilde{\chi}^{-}_{u_o}\big((0, \mu + \pi, a_1)^{-}_{c}\big), \quad a_1 < 0. \tag{4.13}$$

Thus, we need to identify the two points $(0, \mu, a_1)^{+}_{c}$ and $(0, \mu + \pi, a_1)^{-}_{c}$.

4.6. A KS two-fold. We define a space analogous to the standard two-sheeted Riemann surface. It consists of two pieces glued together. We cannot call it two-sheeted because each piece is three-dimensional. We use this space to glue the two branches $\tilde{\chi}^{\pm}_{u_o}$ together. To that end we let $\mathbb{X}^{(2)}_{u_o}$ be the quotient of the disjoint union $\widetilde{\mathbb{X}}^{-}_{u_o} \sqcup \widetilde{\mathbb{X}}^{+}_{u_o}$ when we make the identification

$$\big(\alpha^{-}, \mu^{-}, a_1^{-}\big) \sim \big(\alpha^{+}, \mu^{+}, a_1^{+}\big) \quad \Leftrightarrow \quad \begin{cases} \alpha^{-} = \alpha^{+} = 0, & \text{and}\\ a_1^{-} = a_1^{+} < 0, & \text{and}\\ \mu^{+} = \mu^{-} + \pi. \end{cases}$$

In spherical coordinates

$$\big(r^{-}, \mu^{-}, \nu^{-}\big) \sim \big(r^{+}, \mu^{+}, \nu^{+}\big) \quad \Leftrightarrow \quad \begin{cases} r^{-} = r^{+}, & \text{and}\\ \mu^{+} = \mu^{-} + \pi, & \text{and}\\ \nu^{-} = \nu^{+} = \pi. \end{cases}$$

It is obvious that $\sim$ is an equivalence relation and

$$\mathbb{X}^{(2)}_{u_o} = \big(\widetilde{\mathbb{X}}^{-}_{u_o} \sqcup \widetilde{\mathbb{X}}^{+}_{u_o}\big)/\sim$$

is a real analytic manifold. Moreover,

$$\tilde{\psi}^{-}_{u_o}(\rho_2, \gamma, 0) = (0, \gamma + \pi, a_1)^{-} \sim (0, \gamma, a_1)^{+} = \tilde{\psi}^{+}_{u_o}(\rho_2, \gamma, 0). \tag{4.14}$$

Denote the equivalent class of $(0, \mu, a_1)$, $a_1 < 0$, by $\overline{(0, \mu, a_1)}$.

4.7. The equivalence relation $\sim$ identifies $C^{-}_{u_o}$ and $C^{+}_{u_o}$ after rotating one of them by an angle π, where $C^{\pm}_{u_o}$ are given by (4.10). Therefore we define

$$C_{u_o} = \overline{C^{+}_{u_o}} = \big\{\overline{(0, \mu, a_1)} \bigm| -\pi \leqslant \mu < \pi,\ a_1 < 0\big\}.$$

Then $\mathbb{X}^{(2)}_{u_o}$ is the disjoint union

$$\mathbb{X}^{(2)}_{u_o} = \mathbb{X}^{+}_{u_o} \sqcup C_{u_o} \sqcup \mathbb{X}^{-}_{u_o}.$$

4.8. Now we glue the two maps $\tilde{\psi}^{+}_{u_o}$ and $\tilde{\psi}^{-}_{u_o}$ into one map given by

$$\begin{aligned}
&p_{u_o} : K_{u_o} \to \mathbb{X}^{(2)}_{u_o},\\
&p_{u_o}(u) = \begin{cases} \tilde{\psi}^{\pm}_{u_o}(u) = \psi^{\pm}_{u_o}(u), & u \in K^{\pm}_{u_o},\\ \overline{(0, \gamma, a_1)} = \overline{(0, \gamma, -\rho_1^2)}, & u \in K^{0}_{u_o}.\end{cases}
\end{aligned} \tag{4.15}$$

The map p_{u_o} is well defined by virtue of (4.9) and (4.14).

4.9. The inverse of p_{u_o} is given by

$$
\begin{aligned}
&q_{u_o} : \mathbb{X}^{(2)}_{u_o} \to K_{u_o},\\
&q_{u_o}(y) = \tilde{\chi}^{\pm}_{u_o}\big(x^{\pm}\big), \quad y = \overline{x^{\pm}}\\
&\quad = \begin{cases} \chi^{\pm}_{u_o}(x), & y = \{x\},\ x \in \mathbb{X}^{\pm}_{u_o},\\ \tilde{\chi}^{r}_{u_o}((0, \mu + \delta(r), a_1)^r), & y = \overline{(0, \mu + \delta(r), a_1)} \in C_{u_o}, \end{cases}\\
&r = \pm, \quad \delta(+) = 0, \quad \delta(-) = \pi.
\end{aligned} \tag{4.16}
$$

The identity (4.13) shows that the map q_{u_o} is well defined.

4.10. It follows that p_{u_o} is real bi-analytic with inverse q_{u_o}.

Definition 4.11 *(A 3-d square root map).* We define a square root map by

$$
\begin{aligned}
&\sqrt{x}^{u_o} : \mathbb{X}^{(2)}_{u_o} \to K_{u_o},\\
&\sqrt{x}^{u_o} = q_{u_o}(x)
\end{aligned} \tag{4.17}
$$

where the superscript "u_o" signifies the fact that this square root is defined relative to the u_o-bases. By an obvious abuse of notation we write

$$
\begin{aligned}
&\sqrt{x}^{u_o} : \widetilde{\mathbb{X}}^{+}_{u_o} \to \widetilde{K}^{+}_{u_o},\\
&\sqrt{x}^{u_o} = q_{u_o}(x) = \tilde{\chi}^{+}_{u_o}(x),\\
&-\sqrt{x}^{u_o} : \widetilde{\mathbb{X}}^{-}_{u_o} \to \widetilde{K}^{-}_{u_o},\\
&-\sqrt{x}^{u_o} = q_{u_o}(x) = \tilde{\chi}^{-}_{u_o}(x)
\end{aligned} \tag{4.18}
$$

or

$$
\pm\sqrt{x}^{u_o} = \tilde{\chi}^{\pm}_{u_o}(x).
$$

5. KS cylindrical, torical and spherical coordinates

In this section we define what we call *KS-cylindrical, torical* and *spherical coordinate systems* relative to any fixed but arbitrary u_o by rotating $K^{+}_{u_o}$ about the plane $K^{0}_{u_o}$.

5.1. Any $u \in \mathbb{U}^{\#}_{u_o}$ can be written uniquely in the form $u = L(\tilde{u}_o)\mathbf{s} = \eta + \xi$ given by (3.2). Thus we can define $(\rho_1, \theta), (\rho_2, \lambda), (\rho, \omega), \gamma, z$ and u_+ uniquely as follows:

$$
\begin{aligned}
&(s_1, s_4) = \rho_1(\cos\theta, \sin\theta), \quad \rho_1 > 0,\ \theta \in [-\pi, \pi),\\
&(s_2, s_3) = \rho_2(\cos\lambda, \sin\lambda), \quad \rho_2 > 0,\ \lambda \in [-\pi, \pi),\\
&(\rho_1, \rho_2) = \rho(\cos\omega, \sin\omega), \quad \rho > 0,\ \omega \in [0, \pi/2],\\
&\gamma = \lambda - \theta \mod 2\pi, \quad \gamma \in [-\pi, \pi),\\
&\zeta = A(-\theta)\xi\\
&\quad = L_{23}(\tilde{u}_o)z,\\
&z = (\,z_2 \quad z_3\,)^{\mathsf{T}} = R(-\theta)(\,s_2 \quad s_3\,)^{\mathsf{T}},\\
&u_+ = A(-\theta)u = \rho_1\tilde{u}^{(1)}_o + \zeta.
\end{aligned} \tag{5.1}
$$

Notice that since $K^0_{u_o}$ is invariant under S^1, $\zeta \in K^0_{u_o}$ and $u_+ \in K^+_{u_o}$. When $\theta = 0$, $s_1 = \rho_1$ and when $\theta = -\pi$, $s_1 = -\rho_1$. When $\lambda = 0$, $\rho_2 = |s_2|$.

It follows from (A.34) that

$$\begin{aligned}
\zeta &= \begin{pmatrix} \tilde{u}_o^{(2)} & \tilde{u}_o^{(3)} \end{pmatrix} R(-\theta) \begin{pmatrix} s_2 & s_3 \end{pmatrix}^{\mathsf{T}} \\
&= \rho_2 \big(\tilde{u}_o^{(2)} \cos\gamma + \tilde{u}_o^{(3)} \sin\gamma \big) \\
&= \rho_2 \tilde{u}_o^{(2)\gamma}, \\
|\eta| &= \rho_1, \qquad |\zeta| = |z| = |\xi| = \rho_2.
\end{aligned} \tag{5.2}$$

5.2. Let

$$R(\theta, \lambda) = R_{14}(\theta) \oplus R_{23}(\lambda).$$

Thus,

$$R(\theta, \theta) = B(\theta), \qquad R(\theta + \alpha, \theta + \beta) = B(\theta) R(\alpha, \beta)$$

where $B(\theta)$ satisfies (A.34).

5.3. We can express $u \in \mathbb{U}^{\#}_{u_o}$ uniquely in any of the following forms:

$$\begin{aligned}
u &= L(\tilde{u}_o) \begin{pmatrix} \rho_1 \cos\theta & s_2 & s_3 & \rho_1 \sin\theta \end{pmatrix}^{\mathsf{T}} \\
&= L(\tilde{u}_o) R(\theta, 0) \begin{pmatrix} \rho_1 & s_2 & s_3 & 0 \end{pmatrix}^{\mathsf{T}} \\
&= L(\tilde{u}_o) R(0, \lambda) \begin{pmatrix} s_1 & \rho_2 & 0 & s_4 \end{pmatrix}^{\mathsf{T}} \\
&= L(\tilde{u}_o) R(\theta, \lambda) \begin{pmatrix} \rho_1 & \rho_2 & 0 & 0 \end{pmatrix}^{\mathsf{T}} \\
&= A(\theta) L(\tilde{u}_o) R(0, \gamma) \begin{pmatrix} \rho_1 & \rho_2 & 0 & 0 \end{pmatrix}^{\mathsf{T}} \\
&= \rho A(\theta) L(\tilde{u}_o) R(0, \gamma) \begin{pmatrix} \cos\omega & \sin\omega & 0 & 0 \end{pmatrix}^{\mathsf{T}} \\
&= \rho A(\theta) L(\tilde{u}_o) R(0, \gamma) R_{12}(\omega) \mathbf{b}_1 \\
&= A(\theta) \big(\rho_1 \tilde{u}_o^{(1)} + \rho_2 \tilde{u}_o^{(2)\gamma} \big) \\
&= A(\theta) \big(\rho_1 \tilde{u}_o^{(1)} + \zeta \big) \\
&= A(\theta) u_+.
\end{aligned} \tag{5.3}$$

5.4. It follows from (3.10) that

$$\begin{aligned}
u &= L\big(\tilde{u}_o^{\beta}\big) A(\beta) R(\theta, \lambda) \begin{pmatrix} \rho_1 & \rho_2 & 0 & 0 \end{pmatrix}^{\mathsf{T}} \\
&= L\big(\tilde{u}_o^{\beta}\big) R(\theta - \beta, \lambda + \beta) \begin{pmatrix} \rho_1 & \rho_2 & 0 & 0 \end{pmatrix}^{\mathsf{T}}.
\end{aligned} \tag{5.4}$$

Thus,

$$\begin{aligned}
&(\rho_1)_\beta = \rho_1, \quad (\rho_2)_\beta = \rho_2, \quad \rho_\beta = \rho, \\
&\theta_\beta = \theta - \beta, \quad \lambda_\beta = \lambda + \beta, \quad \gamma_\beta = \lambda_\beta - \theta_\beta = \gamma + 2\beta, \\
&\omega_\beta = \omega, \quad \eta_\beta = \eta, \quad \xi_\beta = \xi, \\
&\zeta_\beta = \zeta^\beta, \quad z_\beta = R(2\beta) z.
\end{aligned} \tag{5.5}$$

5.5. The KS coordinate systems relative to u_o. We define the KS coordinate systems for each fixed but arbitrary u_o as follows:

(1) *The Ks rectangular coordinates*: $(s_1, s_2, s_3, s_4) \in \mathbb{R}^4$.

(2) *KS cylindrical coordinates*: (ρ_1, θ, z), $\rho_1 > 0$, $\theta \in [-\pi, \pi)$ and $z \in \mathbb{R}^2$.
   (a) When it is clear from the context, we write (ρ_1, θ, z) as (ρ_1, θ, ζ).
   (b) Given (ρ_1, θ, z), we have

$$
\begin{aligned}
u &= A(\theta)\big(\rho_1 \tilde{u}_o^{(1)} + z_2 \tilde{u}_o^{(2)} + z_3 \tilde{u}_o^{(3)}\big) \\
&= A(\theta) L(\tilde{u}_o)(\,\rho_1 \quad z_2 \quad z_3 \quad 0\,)^{\mathsf{T}} \\
&= L(\tilde{u}_o) B(\theta)(\,\rho_1 \quad z_2 \quad z_3 \quad 0\,)^{\mathsf{T}} \\
&= L(\tilde{u}_o) \begin{pmatrix} \rho_1 \cos\theta \\ z_2 \cos\theta - z_3 \sin\theta \\ z_2 \sin\theta + z_3 \cos\theta \\ \rho_1 \sin\theta \end{pmatrix}.
\end{aligned} \tag{5.6}
$$

(3) *The KS torical coordinates*: $(\rho_1, \theta, \rho_2, \gamma)$, $\rho_1 > 0$, $\rho_2 > 0$, $\theta \in [-\pi, \pi)$, and $\gamma \in [-\pi, \pi)$.
   (a) Let $\vec{\rho} = (\rho_1, \rho_2)$. Then we can also write the torical coordinates as $(\vec{\rho}, \theta, \gamma)$.
   (b) Given $(\rho_1, \theta, \rho_2, \gamma)$, we have

$$
\begin{aligned}
\zeta &= \rho_2 \tilde{u}_o^{(2)\gamma} = L_{23}(\tilde{u}_o) \begin{pmatrix} \rho_2 \cos\gamma \\ \rho_2 \sin\gamma \end{pmatrix} = L_{23}(\tilde{u}_o) \begin{pmatrix} z_2 \\ z_3 \end{pmatrix}, \\
u &= A(\theta)\big(\rho_1 \tilde{u}_o^{(1)} + \rho_2 \tilde{u}_o^{(2)\gamma}\big) \\
&= \rho_1 \tilde{u}_o^{(1)\theta} + \rho_2 u_o^{(2)^{\theta+\gamma}} \\
&= L(\tilde{u}_o) \begin{pmatrix} \rho_1 \cos\theta \\ \rho_2 \cos(\gamma + \theta) \\ \rho_2 \sin(\gamma + \theta) \\ \rho_1 \sin\theta \end{pmatrix}.
\end{aligned} \tag{5.7}
$$

   Recall that $\lambda = \gamma + \theta \bmod 2\pi$ (5.1).

(4) *The KS spherical coordinates*: $(\rho, \theta, \gamma, \omega)$, $\rho > 0$, $\theta \in [-\pi, \pi)$, $\gamma \in [-\pi, \pi)$, and $\omega \in [0, \pi/2]$.
   Given $(\rho, \theta, \gamma, \omega)$, we have

$$
\begin{aligned}
u &= \rho A(\theta)\big(\tilde{u}_o^{(1)} \cos\omega + \tilde{u}_o^{(2)\gamma} \sin\omega\big) \\
&= \rho\big(\tilde{u}_o^{(1)\theta} \cos\omega + u_o^{(2)^{\theta+\gamma}} \sin\omega\big) \\
&= L(\tilde{u}_o) \begin{pmatrix} \rho \cos\omega \cos\theta \\ \rho \sin\omega \cos(\gamma + \theta) \\ \rho \sin\omega \sin(\gamma + \theta) \\ \rho \cos\omega \sin\theta \end{pmatrix}.
\end{aligned}
$$

5.6. Remarks. The KS cylindrical coordinates $(\rho_1, \theta, z)_{u_o}$ earn their name from the fact that ρ_1 gives the distance from the plane $K_{u_o}^0$, and the angle θ represents a rotation about $K_{u_o}^0$ which can be thought of as a 2-dimensional axis of rotation.

(1) We can think of cylindrical coordinates in $\mathbb{R}^3$ as the restriction of $(\rho_1, \theta, z)_{u_o}$ to 3-dimensional space $\langle \tilde{u}_o^{(1)}, \tilde{u}_o^{(4)}, \tilde{u}_o^{(2)} \rangle$ and use (ρ_1, θ, s_2) with $\tilde{u}_o^{(2)}$ as the axis of rotation[2] and the plane $\langle \tilde{u}_o^{(1)}, \tilde{u}_o^{(4)} \rangle$ as the horizontal plane.[3]
(2) In fact we can also think of cylindrical coordinates in $\mathbb{R}^3$ as a restriction to any 3-dimensional space $\langle \tilde{u}_o^{(1)}, \tilde{u}_o^{(4)}, \tilde{u}_o^{(2)\gamma} \rangle$ (where γ is fixed but arbitrary) and use (ρ_1, θ, ρ_2) with $\tilde{u}_o^{(2)\gamma}$ as the axis of rotation and the plane $\langle \tilde{u}_o^{(1)}, \tilde{u}_o^{(4)} \rangle$ as the horizontal plane. In this case we obtain only the upper half of $\mathbb{R}^3$ because $\rho_2 > 0$.
(3) Another way of interpreting cylindrical coordinates in $\mathbb{R}^3$ is as the coordinates (ρ_2, γ, s_1) on $K_{u_o} = \langle \tilde{u}_o^{(2)}, \tilde{u}_o^{(3)}, \tilde{u}_o^{(1)} \rangle$ with $\tilde{u}_o^{(1)}$ as the axis of rotation.
(4) Polar coordinates in the real plane $\mathbb{R}^2$ can be thought of as the restriction of $(\rho_1, \theta, z)_{u_o}$ or $(\rho_1, \theta, \rho_2, \gamma)$ to the plane $\langle \tilde{u}_o^{(1)}, \tilde{u}_o^{(4)} \rangle$ where $s_2 = s_3 = 0$. In this case we use (ρ_1, θ). Another possibility is restricting $(\rho_1, \theta, \rho_2, \gamma)$ to the plane $K_{u_o}^0 = \langle \tilde{u}_o^{(2)}, \tilde{u}_o^{(3)} \rangle$ where $s_1 = s_4 = 0$ and use (ρ_2, γ).
(5) However, polar coordinates in the complex plane $\mathbb{C}$ are a restriction of spherical coordinates $(\rho, \theta, \gamma, \omega)$ to the plane $\langle \tilde{u}_o^{(1)}, \tilde{u}_o^{(2)} \rangle$ where $s_3 = s_4 = 0$. Recall that the solution to the equation $L(u)u = \tilde{x}_{u_o}^{(1)}$ is the circle $[\tilde{u}_o^{(1)}]$ and to $L(u)u = -\tilde{x}_{u_o}^{(1)}$, the circle $[\tilde{u}_o^{(2)}]$. Therefore if $\tilde{u}_o^{(1)} \simeq 1 \in \mathbb{C}$, we ought to have $\tilde{u}_o^{(2)} \simeq i \in \mathbb{C}$.
(6) Since S^1, S^3 and S^7 are the only parallelizable spheres [2,8], we should be able to investigate the existence of KS cylindrical coordinates in $\mathbb{R}^8$ and their restriction to $\mathbb{R}^n$, $2 \leqslant n \leqslant 8$. The restriction to $2 \leqslant n \leqslant 4$ will yield the ones that we have here. This investigation will be carried out in a different work.

5.7. Since the KS coordinates are defined by rotating the 3-dimensional open half space $K_{u_o}^+$, we should be able to develop similar coordinate systems by rotating $K_{u_o}^-$. We make this more precise presently.

(1) Let $\theta^+ = \theta$ and $z^+ = z$.
(2) If we rotate u by an angle $-t$ we obtain

$$\begin{aligned} u^{-t} &= L(\tilde{u}_o)B(-t)R(\theta, 0)(\,\rho_1 \quad s_2 \quad s_3 \quad 0\,)^\mathsf{T} \\ &= L(\tilde{u}_o)R(\theta - t, -t)(\,\rho_1 \quad s_2 \quad s_3 \quad 0\,)^\mathsf{T}. \end{aligned} \tag{5.8}$$

(3) It follows that $u^{-t} \in K_{u_o}^+$ iff $t = \theta \bmod 2\pi$. This is another way to see that θ is the unique angle in $[-\pi, \pi)$ such that $u^{-\theta} \in K_{u_o}^+$.
(4) Let $\theta^- = (\theta - \pi) \bmod 2\pi$. Since $A(\pi) = -I$, and since $B(\beta)$ satisfies (A.34), if we take $t = \theta^-$ we obtain

$$\begin{aligned} A\bigl(-\theta^-\bigr)u &= L(\tilde{u}_o)B(-\theta^-)R(\theta, 0)(\,\rho_1 \quad s_2 \quad s_3 \quad 0\,)^\mathsf{T} \\ &= L(\tilde{u}_o)R\bigl(\pi, -\theta^+ + \pi\bigr)(\,\rho_1 \quad s_2 \quad s_3 \quad 0\,)^\mathsf{T} \\ &= -L(\tilde{u}_o)R\bigl(0, -\theta^+\bigr)(\,\rho_1 \quad s_2 \quad s_3 \quad 0\,)^\mathsf{T} \\ &= -u_+ \\ &= -\bigl(\rho_1\tilde{u}_o^{(1)} + \zeta\bigr) \in K_{u_o}^-. \end{aligned} \tag{5.9}$$

[2] Standard z-axis.
[3] Standard xy-plane.

And we can define

$$\theta^- = (\theta - \pi) \quad \bmod 2\pi, \qquad u_- = -u_+, \qquad z^- = -z^+.$$

(5) Thus $[u] \cap K^{\pm}_{u_o} = \{u_\pm\}$ where $u_\pm := A(-\theta^\pm)u$.

The following two corollaries are immediate consequences of the discussion we started in Section 5.1.

Corollary 5.8.

(1) *The following two maps are well defined*:

$$\phi^{\pm}_{u_o} : \mathbb{U}^{\#}_{u_o} \to [-\pi, \pi),$$
$$\phi^{\pm}_{u_o}(u) = \theta^{\pm}.$$

It follows that

$$u_\pm := A\big(-\phi^{\pm}_{u_o}(u)\big)u \in K^{\pm}_{u_o},$$
$$u_- = -u_+ = A(\pi)u_+,$$
$$u = A\big(\phi^{\pm}_{u_o}(u)\big)u_\pm,$$
$$\phi^{+}_{u_o}\big(u^t\big) = \phi^{+}_{u_o}(u) + t,$$
$$\phi^{+}_{u_o^{\alpha}}(u) = \phi^{+}_{u_o}(u) - \alpha.$$

(2) *Since the flow of the vector field* (2.5) *is real analytic, it follows that* $\phi^{\pm}_{u_o} : \mathbb{U}^{\#}_{u_o} \to [-\pi, \pi)$ *are real analytic and for any* $u \in \mathbb{U}^{\#}_{u_o}$, *the restriction*

$$\phi^{+}_{u_o}\lfloor[u] : [u] \to S^1$$

is real bi-analytic.

Corollary 5.9. *Let a square cup* $\sqcup$ *stands for disjoint union. Then*

$$\mathbb{U}^{\#}_{u_o} = \bigsqcup_t A(t)K^{+}_{u_o} = \bigsqcup_t K^{+}_{u_o^t}.$$

Let $\pi_{u_o}(u) = A(-\phi^+(u))u$. *It follows that* $\mathcal{P}^{\#}_{u_o} = (\mathbb{U}^{\#}_{u_o}, \pi_{u_o}, K^{+}_{u_o}, G)$ *is a trivial principal bundle.*

Proof. Lemma 2.9 tells us that $K^{+}_{u_o^t} = A(t)K^{+}_{u_o}$ and that $K^{0}_{u_o^t} = K^{0}_{u_o}$. $\square$

Lemma 5.10.

(1) *The flow of the real analytic vector field* (2.5) *provides a real bi-analytic map from* $K^{+}_{u_o} \times G$ *onto* $\mathbb{U}^{\#}_{u_o}$ *which we denote by*

$$F_{u_o} : K^{+}_{u_o} \times G \to \mathbb{U}^{\#}_{u_o},$$
$$F_{u_o}(u, t) = A(t)u,$$
$$F^{-1}_{u_o}(u) = \big(A\big(-\phi^+(u)\big)u, \phi^+(u)\big). \tag{5.10}$$

(2) *The flow F_{u_o} induces a real bi-analytic section map from $K^+_{u_o^\alpha}$ to $K^+_{u_o^\beta}$ which we denote by*

$$F^{\alpha,\beta}_{u_o} : K^+_{u_o^\alpha} \to K^+_{u_o^\beta}, \quad -\pi \leqslant \alpha,\ \beta < \pi. \tag{5.11}$$

(3) *The flow is transversal to $K^\pm_{u_o}$.*
(4) *For any $u \in K^\pm_{u_o}$, $\pm u^{(4)} \cdot \tilde{u}^{(4)}_o = \pm s_1 > 0$.*

Proof. Parts (1) and (2) follow immediately from Corollary 5.9. As for parts (3) and (4), write $u \in K^\pm_{u_o}$ as $u = s_1\tilde{u}^{(1)}_o + z_2\tilde{u}^{(2)}_o + z_3\tilde{u}^{(3)}_o$, $\rho_1 > 0$. Thus,

$$\pm u^{(4)} \cdot \tilde{u}^{(4)}_o = \pm u \cdot \tilde{u}^{(1)}_o = \pm s_1 \left|\tilde{u}^{(1)}_o\right|^2 = \rho_1 > 0. \qquad \square$$

Proposition 5.11. *$\mathcal{P} = (\mathbb{U}^*, \psi, \mathbb{X}^*, G)$ is a real analytic principal bundle.*

Proof. Corollary 5.9 tells us that $\mathbb{U}^\#_{u_o} = \bigsqcup_t A(t)K^+_{u_o}$. Lemma 5.10 tells us that the flow (5.10) is real bi-analytic. The map $\psi^+_{u_o} : K^+_{u_o} \to \mathbb{X}^+_{u_o}$ is real bi-analytic. Thus we have a real bi-analytic local trivialization of $\mathcal{P}$ near any point $u \notin K^0_{u_o}$ given by

$$\begin{aligned}
&\tau_{u_o} : \psi^{-1}\big(\mathbb{X}^+_{u_o}\big) = \mathbb{U}^\#_{u_o} \overset{F_{u_o}}{\Longleftrightarrow} K^+_{u_o} \times G \overset{(\psi^+_{u_o},\mathrm{id})}{\Longleftrightarrow} \mathbb{X}^+_{u_o} \times G,\\
&u \overset{F_{u_o}}{\longleftrightarrow} (u_+, \theta) \overset{\psi^+_{u_o}}{\longleftrightarrow} \big(\psi^+_{u_o}(u_+), \theta\big).
\end{aligned}$$

For points in $K^0_{u_o}$ we use τ_{w_o} with any $w_o \notin [u_o]$. For example take $w_o = u^{(2)}_o$. In this case $K^0_{w_o} = S_{u_o} = \mathrm{span}\{u^{(1)}_o, u^{(4)}_o\}$. Thus $K^0_{w_o} \cap K^0_{u_o} = \emptyset$. Let $u = L(\tilde{u}_o)\mathbf{s}_{u_o} = L(\widetilde{w}_o)\mathbf{s}_{w_o}$. Then

$$\begin{aligned}
u &= L(\tilde{u}_o)\mathbf{s}_{u_o} = L(\widetilde{w}_o)\mathbf{s}_{w_o}\\
&= L\big(\tilde{u}^{(2)}_o\big)\mathbf{s}_{w_o} = L(\tilde{u}_o)I_2\mathbf{s}_{w_o}.
\end{aligned}$$

Thus the transition function between the two trivializations is given by

$$\mathbf{s}_{w_o} = -I_2\mathbf{s}_{u_o}. \qquad \square$$

Definition 5.12 *(The horizontal bundle $H\mathbb{U}^*$).* Let

$$\begin{aligned}
H\mathbb{U}^* &:= \bigcup_{u\in\mathbb{U}^*} H_u\mathbb{U}^*,\\
H_u\mathbb{U}^* &:= \big\{(u, v) \in T_u\mathbb{U}^* \bigm| v \in \mathbb{V},\ \ell(u, v) = 0\big\}\\
&\cong \big\{u^{(4)}\big\}^\perp = K_u.
\end{aligned}$$

$H\mathbb{U}^*$ is called the horizontal sub-bundle of the principal bundle $\mathcal{P} = (\mathbb{U}^*, \psi, \mathbb{X}^*, G)$. It is also called the principal connection of $\mathcal{P}$.

A vector field $V : \mathbb{U}^* \to T\mathbb{U}^*$ is called horizontal iff $V(u) \in H_u\mathbb{U}^*$ for all $u \in \mathbb{U}^*$.

Proposition 5.13. *In view of Proposition* 2.10*, the horizontal bundle $H\mathbb{U}^*$ is not integrable. Equivalently the vector fields $\langle u^{(1)}, u^{(2)}, u^{(3)}\rangle$ are not in involution.*

5.14. Planes of Levi-Civita type [14]. A plane $P = \mathrm{span}\{u, v\}$ is called a *plane of Levi-Civita type* or an L-plane for short iff $\ell(u, v) = 0$.

Let

$$P_{u_o}^{\gamma} = \operatorname{span}\{\tilde{u}_o^{(1)}, \tilde{u}_o^{(2)\gamma}\}, \quad \gamma \in [0, \pi).$$

It follows that $P_{u_o}^{\gamma}$ is an L-plane. In fact any L-plane $P = \operatorname{span}\{u, v\}$ is of the form P_u^{γ} for some γ because in this case $v \in K_u$ and hence can be written in the form $s_1 u^{(1)} + \rho_2 u^{(2)\gamma}$ for some $\gamma \in [-\pi, \pi)$. If $\gamma > \pi$, replace it by $\gamma - \pi$. It follows that

$$P_{u_o}^{\gamma} = P_{u_o}^{\gamma+\pi}, \qquad K_{u_o} \sqcup \{0\} = \bigcup_{\gamma=0}^{\pi} P_{u_o}^{\gamma}.$$

For each L-plane $P_{u_o}^{\gamma}$ we define a corresponding plane $Q_{u_o}^{\gamma}$ in $\mathbb{X}_{u_o}^*$ by

$$Q_{u_o}^{\gamma} = \operatorname{span}\{\tilde{x}_{u_o}^{(1)}, \tilde{x}_{u_o}^{(2)\cdot\gamma}\}, \quad \gamma \in [0, \pi).$$

We also have

$$Q_{u_o}^{\gamma} = Q_{u_o}^{\gamma+\pi}, \qquad \mathbb{X}_{u_o}^* \sqcup \{0\} = \bigcup_{\gamma=0}^{\pi} Q_{u_o}^{\gamma}.$$

We will see soon that the KS map furnishes a double cover from $P_{u_o}^{\gamma}$ to $Q_{u_o}^{\gamma}$ that coincides with the standard squaring map on $\mathbb{C}$.

5.15. The KS map in KS coordinates. Recall that in $\mathbb{U}_{u_o}^{\#}$ $\gamma \in [-\pi, \pi)$ and $\kappa = \omega \in [0, \pi/2)$ and we write

$$\begin{aligned}
u &= A(\theta)u_+ = A(\theta)\big(\rho_1 \tilde{u}_o^{(1)} + \rho_2 \tilde{u}_o^{(2)\gamma}\big), \quad u \notin K_{u_o}^0 \\
&= \rho A(\theta)\big(\tilde{u}_o^{(1)} \cos\omega + \tilde{u}_o^{(2)\gamma} \sin\omega\big), \\
u &= \rho_2 \tilde{u}_o^{(2)\gamma}, \quad u \in K_{u_o}^0, \\
x &= a_1 \tilde{x}_{u_o}^{(1)} + \alpha \tilde{x}_{u_o}^{(2)\cdot\mu}.
\end{aligned}$$

Moreover, we know that $\psi(u) = \psi(u_+) = \psi_{u_o}^+(u_+)$. It follows from (5.5) with $\beta = \theta$ and from (4.7) with $\kappa = \omega$ that

$$\begin{aligned}
\psi(u) &= L(u)u \\
&= L(u^+)u^+, \\
\psi_{u_o}^+\big(u^+\big) &= x \\
&= \big(\rho_1^2 - \rho_2^2\big)\tilde{x}_{u_o}^{(1)} + 2\rho_1\rho_2 \tilde{x}_{u_o}^{(2)\cdot\gamma} \\
&= \rho^2\big(\tilde{x}_{u_o}^{(1)} \cos 2\omega + \tilde{x}_{u_o}^{(2)\cdot\gamma} \sin 2\omega\big), \\
a_1 &= \rho_1^2 - \rho_2^2 = \rho^2 \cos 2\omega, \\
a_2 &= 2\rho_1\rho_2 \cos\gamma = \rho \sin 2\omega \cos\gamma, \\
a_3 &= 2\rho_1\rho_2 \sin\gamma = \rho \sin 2\omega \sin\gamma, \\
\alpha &= 2\rho_1\rho_2, \\
\mu &= \gamma = \lambda - \theta \mod 2\pi, \\
r &= \rho^2, \\
\nu &= 2\omega.
\end{aligned} \tag{5.12}$$

We can also relate $(\rho, \rho_1, \rho_2, \gamma, \omega)$ to the variables $(r, a_1, \alpha, \mu, \nu)$ in the following form:

$$\begin{aligned} re^{i\nu} &= a_1 + i\alpha = (\rho_1 + i\rho_2)^2 \\ &= \rho_1^2 - \rho_2^2 + 2\rho_1\rho_2 i \\ &= \rho^2 e^{2\omega i}, \\ \mu &= \gamma. \end{aligned} \tag{5.13}$$

Since $\rho_2 > 0$, we can invert (5.13) when $x \notin \mathbb{R}^-_{x_o}$ and obtain

$$\begin{aligned} u_+ \simeq \left(\rho e^{i\omega}, \gamma\right) = (\rho_1 + i\rho_2, \gamma) &= \left(\sqrt{a_1 + i\alpha}, \mu\right), \quad \text{when } x \notin \mathbb{R}^-_{x_o} \\ &= \left(\sqrt{r}e^{i\nu/2}, \mu\right). \end{aligned} \tag{5.14}$$

When $x \in \mathbb{R}^-_{x_o}$ we have to make a choice, say

$$u_+ = \sqrt{|a_1|}\,\tilde{u}_o^{(1)}. \tag{5.15}$$

In both cases we obtain the fiber or orbit

$$\psi^{-1}(x) = \left\{A(\theta)u_+ \;\middle|\; -\pi \leqslant \theta < \pi\right\}. \tag{5.16}$$

6. The full square root map

The square root map defined by (4.16) cannot be the full square root map because we know that $\psi^{-1}(x)$ is always a circle, while $\mathbb{X}^{(2)}_{u_o}$ has a circle above every $x \in \mathbb{R}^-_{x_o}$ but only two points above every $x \notin \mathbb{R}^-_{x_o}$. In other words, with $\hat{\psi}_{u_o}$ given by (3.15), when $x \notin \mathbb{R}^-_{x_o}$, $\hat{\psi}^{-1}_{u_o}(x)$ is two points, but when $x \in \mathbb{R}^-_{x_o}$, $\hat{\psi}^{-1}_{u_o}(x)$ is a full circle contained in $K^0_{u_o}$.

To define the full square root map we need to rotate the branch $\chi^+_{u_o} : \mathbb{X}^+_{u_o} \to K^+_{u_o}$. In Section 5 we saw how to rotate $K^+_{u_o}$ to obtain $\mathbb{U}^*_{u_o} = \bigsqcup_\beta K^+_{u_o^\beta}$. Therefore we need to work with $\mathbb{X}^+_{u_o^\beta} \times S^1$, but then we need to use the rotated basis $\{\tilde{x}^{(2)}_{u_o^\theta}, \tilde{x}^{(3)}_{u_o^\theta}, \tilde{x}^{(1)}_{u_o^\theta}\}$ and relate it to the basis $\langle \tilde{x}^{(2)}_{u_o}, \tilde{x}^{(3)}_{u_o}, \tilde{x}^{(1)}_{u_o}\rangle$. We blew up $\mathbb{R}^-_{x_o}$ in both $\mathbb{X}^\pm_{u_o}$ to $C^\pm_{u_o}$ given by (4.10). We need to do that in cylindrical coordinates.

6.1. It is possible to use cylindrical coordinates to blow up only the deleted negative x_o-axis in $\mathbb{X}^\pm_{u_o}$, that is $\mathbb{R}^-_{x_o}$, because the origin is not included in $\mathbb{X}^\pm_{u_o}$. First we write $\mathbb{X}^\pm_{u_o}$ as the disjoint union

$$\mathbb{X}^+_{u_o} \simeq \mathbb{X}^-_{u_o} \simeq (0,\infty) \times S^1 \times (-\infty, 0) \sqcup (0,\infty) \times S^1 \times \{0\} \sqcup \mathbb{R}^2 \times (0,\infty) \tag{6.1}$$

where the variables in the first and second parts are (α, μ, a_1). In the third part we use (a_2, a_3, a_1). We also write $C^\pm_{u_o}$ in the form

$$C^+_{u_o} \simeq C^-_{u_o} \simeq \{0\} \times S^1 \times (-\infty, 0). \tag{6.2}$$

Thus

$$\begin{aligned} \widetilde{\mathbb{X}}^+_{u_o} &= \mathbb{X}^+_{u_o} \sqcup C^+_{u_o} \simeq \mathbb{X}^-_{u_o} \sqcup C^-_{u_o} = \widetilde{\mathbb{X}}^-_{u_o}, \\ \widetilde{\mathbb{X}}^+_{u_o} &\simeq \widetilde{\mathbb{X}}^-_{u_o} \simeq [0,\infty) \times S^1 \times (-\infty, 0) \sqcup (0,\infty) \times S^1 \times \{0\} \sqcup \mathbb{R}^2 \times (0,\infty). \end{aligned}$$

We are going to write points in $C^+_{u_o}$ and $C^-_{u_o}$ in either of the forms

$$(0,\mu,a_1)=\big(a_1\tilde{x}^{(1)}_{u_o},\mu\big),\quad a_1\in(-\infty,0),\ \mu\in[-\pi,\pi).$$

We will denote points in $\widetilde{\mathbb{X}}^-_{u_o}$ and $\widetilde{\mathbb{X}}^+_{u_o}$ by $x^-=(\alpha^-,\mu^-,a_1^-)$ and $x^+=(\alpha^+,\mu^+,a_1^+)$.

6.2. Rotating the orthonormal basis $\langle\tilde{x}^{(2)}_{u_o},\tilde{x}^{(3)}_{u_o},\tilde{x}^{(1)}_{u_o}\rangle$. Notice that $\mathbb{X}^+_{u_o^\beta}$ has the description given by (6.1) and is equipped with the orthonormal basis $\{\tilde{x}^{(2)}_{u_o^\beta},\tilde{x}^{(3)}_{u_o^\beta},\tilde{x}^{(1)}_{u_o^\beta}\}$ and that $\tilde{x}^{(1)}_{u_o^\beta}=\tilde{x}^{(1)}_{u_o}$. The open upper half space $K^+_{u_o^\beta}$ is equipped with the orthonormal basis $\{\tilde{u}_o^{\beta(2)},\tilde{u}_o^{\beta(3)},\tilde{u}_o^{\beta(1)}\}$.

It follows from (A.38), (5.12), and (4.3) that the rotating orthonormal basis $\langle\tilde{x}^{(2)}_{u_o^\beta},\tilde{x}^{(3)}_{u_o^\beta},\tilde{x}^{(1)}_{u_o^\beta}\rangle$ is given by

$$\begin{aligned}
\tilde{x}^{(2)}_{u_o^\beta}&=\tilde{x}^{(2)\cdot(-2\beta)}_{u_o},\\
\tilde{x}^{(3)}_{u_o^\beta}&=\tilde{x}^{(3)\cdot(-2\beta)}_{u_o}\\
&=\tilde{x}^{(2)\cdot(-2\beta+\pi/2)}_{u_o},\\
\tilde{x}^{(1)}_{u_o^\beta}&=\tilde{x}^{(1)}_{u_o}=\tilde{x}_{u_o}.
\end{aligned}\tag{6.3}$$

It follows from (6.3) and (4.4) that if x^β is x but represented in the β-orthonormal frame $\langle\tilde{x}^{(2)}_{u_o^\beta},\tilde{x}^{(3)}_{u_o^\beta},\tilde{x}^{(1)}_{u_o^\beta}\rangle$, we have

$$\begin{aligned}
x^\beta&=C\big(\tilde{x}_{u_o^\beta}\big)\mathbf{a}_\beta\\
&=\big(\,\tilde{x}^{(1)}_{u_o^\beta}\quad\tilde{x}^{(2)}_{u_o^\beta}\quad\tilde{x}^{(3)}_{u_o^\beta}\,\big)\mathbf{a}^\beta\\
&=\big(\,\tilde{x}^{(1)}_{u_o}\quad\tilde{x}^{(2)\cdot(-2\beta)}_{u_o}\quad\tilde{x}^{(3)\cdot(-2\beta)}_{u_o}\,\big)\mathbf{a}_\beta\\
&=C(\tilde{u}_o)(1\oplus R_{-2\beta})\mathbf{a}^\beta\\
&=C(\tilde{u}_o)\mathbf{a}\\
&=x,\\
1\oplus R_\lambda&=\begin{pmatrix}1&0&0\\0&\cos\lambda&-\sin\lambda\\0&\sin\lambda&\cos\lambda\end{pmatrix}.
\end{aligned}\tag{6.4}$$

Thus

$$\begin{aligned}
&\mathbf{a}_\beta=(1\oplus R_{2\beta})\mathbf{a},\\
&(r_\beta,\nu_\beta,a_{1\beta},\alpha_\beta)=(r,\nu,a_1,\alpha),\quad \mu_\beta=\mu+2\beta,\\
&(\alpha,\gamma+2\theta,a_1)_{u_o^\theta}\simeq a_1\tilde{x}^{(1)}_{u_o^\theta}+\alpha\tilde{x}^{(2)(\mu+2\theta)}_{u_o^\theta}\\
&\qquad\qquad\qquad\qquad=a_1\tilde{x}^{(1)}_{u_o}+\alpha\tilde{x}^{(2)\cdot\mu}_{u_o}\simeq(\alpha,\gamma,a_1)_{u_o}.
\end{aligned}\tag{6.5}$$

Recall also (5.5).

Notice that since we are working with $\widetilde{K}^+_{u_o}$ and $\widetilde{K}^+_{u_o^\theta}$, $s_1\geqslant 0$ and $\kappa=\omega\in[0,\pi/2]$. Also recall (3.11).

6.3. We know that $L(u^\beta)u^\beta = L(u)u = L(u_+)u_+$. Thus

$$\tilde\psi^+_{u_o^\theta}\big(u^\theta\big) = L\big(u^\theta\big)u^\theta = L(u)u = \tilde\psi^+_{u_o}(u), \quad u \in K_{u_o}.$$

We can also simplify the left-hand side to the right-hand side using (4.4), (3.19), and (A.37):

$$\begin{aligned}
\tilde\psi^+_{u_o^\theta}\big(u^\theta\big) &= L\big(u^\theta\big)u^\theta, \quad u \in K^+_{u_o} \\
&= \big(s_1^2 - \rho_2^2\big)\tilde x^{(1)}_{u_o^\theta} + 2s_1\rho_2 L\big(\tilde u_o^\theta\big)\tilde u_o^{\theta\cdot(2)\cdot(\gamma+2\theta)} \\
&= \big(s_1^2 - \rho_2^2\big)\tilde x^{(1)}_{u_o^\theta} + 2s_1\rho_2 L(\tilde u_o)\tilde u_o^{\theta\cdot(2)\cdot(\gamma+\theta)} \\
&= \big(s_1^2 - \rho_2^2\big)\tilde x^{(1)}_{u_o} + 2s_1\rho_2 \tilde x_o^{(2)\cdot\gamma} \\
&= L(u)u = \tilde\psi^+_{u_o}(u) = x
\end{aligned} \tag{6.6}$$

which agrees with (6.5). It follows that

$$\begin{aligned}
&\tilde\psi^+_{u_o^\theta} : \widetilde K^+_{u_o^\theta} \to \widetilde{\mathbb X}^+_{u_o^\theta}, \\
&\tilde\psi^+_{u_o^\theta}(u) = \begin{cases} a_1\tilde x^{(1)}_{u_o^\theta} + \alpha\tilde x^{(2)(\gamma+2\theta)}_{u_o^\theta}, & s_1 > 0, \\ (0, \gamma+2\theta, a_1)_{u_o^\theta}, & s_1 = 0, \end{cases} \\
&\qquad = \begin{cases} a_1\tilde x^{(1)}_{u_o^\theta}, & \rho_2 = 0, \\ (\alpha, \gamma+2\theta, a_1)_{u_o^\theta}, & \rho_2 > 0, \end{cases} \\
&\qquad = \begin{cases} a_1\tilde x^{(1)}_{u_o^\theta}, & \rho_2 = 0, \\ (\alpha, \gamma+2\theta, a_1)_{u_o^\theta}, & \rho_2 > 0, s_1 > 0, \\ (0, \gamma+2\theta, a_1)_{u_o^\theta}, & s_1 = 0, \end{cases} \\
&a_1 + i\alpha = (s_1 + i\rho_2)^2, \quad \rho_2 \geqslant 0.
\end{aligned} \tag{6.7}$$

6.4. The inverse of $\tilde\psi^+_{u_o^\theta}$ is given by rotating $\widetilde{\mathbb X}^+_{u_o}$ (4.12), (4.8):

$$\begin{aligned}
&\tilde\chi^+_{u_o^\theta} : \widetilde{\mathbb X}^+_{u_o^\theta} \to \widetilde K^+_{u_o^\theta}, \\
&\tilde\chi^+_{u_o^\theta}\big(x^\theta\big) = A(\theta)\tilde\chi^+_{u_o}(x) \\
&\qquad = A(\theta)\big(s_1\tilde u_o^{(1)} + \rho_2\tilde u_o^{(2)\mu}\big) \\
&\qquad = s_1\tilde u_o^{\theta(1)} + \rho_2\tilde u_o^{\theta(2)(\mu+2\theta)} \\
&\qquad = \begin{cases} s_1\tilde u_o^{\theta(1)}, & x = a_1\tilde x^{(1)}_{u_o}, a_1 > 0, \\ (\rho_2, \mu+2\theta, s_1)_{u_o^\theta}, & x = (\alpha, \mu, a_1), \alpha > 0, \\ (\rho_2, \mu+2\theta, 0)_{u_o^\theta}, & x = (0, \mu, a_1), a_1 < 0, \end{cases} \\
&s_1 + i\rho_2 = \sqrt{a_1 + i\alpha}, \quad \rho_2 \geqslant 0.
\end{aligned} \tag{6.8}$$

Recall that $\mu_\theta = \mu + 2\theta$ as given by (6.5).

6.5. Let

$$\widetilde{\mathbb X}^\theta_{u_o} := \widetilde{\mathbb X}^+_{u_o} \times \{\theta\} \simeq \widetilde{\mathbb X}^+_{u_o^\theta}.$$

The main difference is that $\widetilde{\mathbb{X}}^+_{u_o}$ is equipped with the orthonormal basis $\langle \tilde{x}^{(2)}_{u_o}, \tilde{x}^{(3)}_{u_o}, \tilde{x}^{(1)}_{u_o}\rangle$ while $\widetilde{\mathbb{X}}^+_{u_o^\theta}$ is equipped with the orthonormal basis $\{\tilde{x}^{(2)}_{u_o^\theta}, \tilde{x}^{(3)}_{u_o^\theta}, \tilde{x}^{(1)}_{u_o^\theta}\}$. Thus $\widetilde{\mathbb{X}}^\theta_{u_o}$ is equipped with the basis $\langle \tilde{x}^{(2)}_{u_o}, \tilde{x}^{(3)}_{u_o}, \tilde{x}^{(1)}_{u_o}\rangle$. Recall that $\tilde{x}^{(1)}_{u_o^\theta} = \tilde{x}^{(1)}_{u_o}$. We write

$$(\alpha, \gamma, a_1; \theta) = \left(a_1 \tilde{x}^{(1)}_{u_o} + \alpha \tilde{x}^{(2)\cdot\gamma}_{u_o}; \theta\right) \in \widetilde{\mathbb{X}}^\theta_{u_o}.$$

6.6. Recall the definition of $\phi^+_{u_o}$ given in Corollary 5.8 and notice that if $u \in \widetilde{K}^+_{u_o^\theta} \setminus K^0_{u_o^\theta}$ then $\phi^+_{u_o}(u) = \theta$. But θ and $\phi^+_{u_o}$ are not defined for $u \in K^0_{u_o^\theta}$. In fact

$$K^0_{u_o^\theta} = K^0_{u_o}, \qquad \bigcap_\theta \widetilde{K}^+_{u_o^\theta} = K^0_{u_o}.$$

The difference between the different $\widetilde{K}^+_{u_o^\theta}$'s is the bases. And we cannot glue together the collection of maps $\{\tilde{\psi}^+_{u_o^\theta}\}$ because they do not agree on $K^0_{u_o}$.

6.7. Define the following

$$\begin{aligned}
&\pi^\theta_{u_o} : \widetilde{K}^+_{u_o^\theta} \to \widetilde{\mathbb{X}}^\theta_{u_o}, \\
&\pi^\theta_{u_o}(u) = \left(\tilde{\psi}^+_{u_o}\left(u^{-\theta}\right); \theta\right)
\end{aligned} \tag{6.9}$$

where $\tilde{\psi}^+_{u_o}$ is given by (4.11). The inverse of $\pi^\theta_{u_o}$ is given by

$$\begin{aligned}
&\xi^\theta_{u_o} : \widetilde{\mathbb{X}}^\theta_{u_o} \to K^+_{u_o^\theta}, \quad \theta \in [0, 2\pi), \\
&\xi^\theta_{u_o}\left((x, \theta)\right) = A(\theta)\tilde{\chi}^+_{u_o}(x) = \tilde{\chi}^+_{u_o^\theta}\left(x^\theta\right)
\end{aligned}$$

where $\tilde{\chi}^+_{u_o}(x)$ is given by (4.12). Here $A(\theta)$ plays the role of the $\pm$ sign in front of the standard square root. Notice that

$$\begin{aligned}
&\tilde{\psi}^+_{u_o} = \pi^0_{u_o}, \qquad \tilde{\chi}^+_{u_o} = \xi^0_{u_o}, \\
&\tilde{\psi}^-_{u_o} = \pi^\pi_{u_o}, \qquad \tilde{\chi}^-_{u_o} = \xi^\pi_{u_o}.
\end{aligned} \tag{6.10}$$

Lemma 6.8. *Let $y = (x, \theta) = (0, \mu, a_1, \theta)$ and $y' = (x', \theta') = (0, \mu', a_1, \theta')$ with $a_1 < 0$ and $\mu + \theta = \mu' + \theta'$. Then*

$$\xi^\theta_{u_o}(x, \theta) = \xi^{\theta'}_{u_o}(x', \theta'), \quad \mu + \theta = \mu' + \theta'. \tag{6.11}$$

Proof.

$$\begin{aligned}
\xi^\theta_{u_o}(x, \theta) &= A(\theta)\tilde{\chi}^+_{u_o}(x) = \rho_2 \tilde{u}_o^{(2)(\mu+\theta)} \\
&= \rho_2 \tilde{u}_o^{(2)(\mu'+\theta')} = A(\theta')\tilde{\chi}^+_{u_o}(x) \\
&= \xi^{u_o}_{\theta'}(x', \theta'). \qquad \square
\end{aligned}$$

The identity (6.11) suggests that we need to identify the two points y and y' somehow.

6.9. The space. Define a relation on $\widetilde{\mathbb{X}}_{u_o}^+ \times S^1$ by

$$(\alpha, \mu, a_1; \theta) \sim (\alpha', \mu', a_1'; \theta') \quad \Leftrightarrow \quad \begin{cases} \alpha = \alpha' = 0 \quad \text{and} \\ a_1 = a_1' < 0 \quad \text{and} \\ \theta + \mu = \theta' + \mu'. \end{cases} \tag{6.12}$$

Let

$$\mathfrak{X}_{u_o}^* = \left(\widetilde{\mathbb{X}}_{u_o}^+ \times S^1\right)/\sim.$$

It is obvious that $\sim$ is an equivalence relation and that $\mathfrak{X}_{u_o}^*$ is a real analytic manifold. The equivalent classes are given by

$$\overline{(x,\theta)} = \begin{cases} \{(x,\theta)\}, & x \in \mathbb{X}_{u_o}^+ = \widetilde{\mathbb{X}}_{u_o}^+ \setminus C_{u_o}^+, \\ \{(0,\gamma', a_1, \theta') \mid \gamma' + \theta' = \gamma + \theta\}, & x = (0, \gamma, a_1) \in C_{u_o}^+ \end{cases}$$

where $C_{u_o}^+$ is given by (4.10) and (6.2). Unless it leads to ambiguity, we write (x,θ) for $\overline{(x,\theta)}$ when $x \in \mathbb{X}_{u_o}^+$.

6.10. The squaring map. Recall that for any $u \notin K_{u_o}^0$ there is a unique $\theta = \phi^+(u) \in [0, 2\pi)$ such that $A(-\theta)u \in K_{u_o}^+$, where ϕ^+, the real analytic function given in Corollary 5.8.

Define the squaring map associated with u_o by

$$\begin{aligned}
&\Psi^{u_o} : \mathbb{U}_{u_o}^* \to \mathfrak{X}_{u_o}^*, \\
&\Psi^{u_o}(u) = \begin{cases} \pi_{u_o}^\theta(u) = (\tilde{\psi}_{u_o}^+(u^{-\theta}); \theta), & u \in \mathbb{U}^* \setminus K_{u_o}^0, \\ \overline{(0,\gamma,a_1;0)}, & u \in K_{u_o}^0, \end{cases} \\
&\phantom{\Psi^{u_o}(u)} = \begin{cases} (\alpha,\gamma,a_1;\theta), & u \in \mathbb{U}^* \setminus K_{u_o}^0, \\ \overline{(0,\gamma,a_1;0)}, & u \in K_{u_o}^0, \end{cases} \\
&\phantom{\Psi^{u_o}(u)} = \begin{cases} (a_1 \tilde{x}_{u_o}^{(1)}; \theta), & \rho_2 = 0 \ (\Rightarrow s_1 > 0), \\ (\alpha,\gamma,a_1;\theta), & \rho_2 \neq 0,\ s_1 > 0, \\ \overline{(0,\gamma,a_1;0)}, & s_1 = 0 \ (\Rightarrow \rho_2 > 0), \end{cases} \\
&\theta = \phi_+(u), \quad u \in \mathbb{U}^* \setminus K_{u_o}^0, \\
&a_1 + i\alpha = (s_1 + i\rho_2)^2, \quad s_1 \geqslant 0.
\end{aligned} \tag{6.13}$$

6.11. The full square root. First we write $\mathfrak{X}_{u_o}^*$ as the disjoint union

$$\begin{aligned}
&\mathfrak{X}_{u_o}^* = \left(\mathbb{X}_{u_o}^+ \times S^1\right) \sqcup \mathfrak{C}_{u_o}, \\
&\mathfrak{C}_{u_o} = \left(\{0\} \times \mathbb{R}_{x_o}^- \times S^1 \times S^1\right)/\sim.
\end{aligned} \tag{6.14}$$

In other words $\mathfrak{C}_{u_o}$ is the quotient of the part of $\widetilde{\mathbb{X}}_{u_o}^+ \times S^1$ that is affected by the equivalence relation $\sim$.

We define the *full square root* map as the inverse of Ψ^{u_o} which is given by rotating $\chi_{u_o}^+$ (4.12), (4.13) and (4.8). Thus the inverse is given by

$$\begin{aligned}
&\Xi^{u_o} : \left(\mathbb{X}_{u_o}^+ \times S^1\right) \sqcup \mathfrak{C}_{u_o} \to \mathbb{U}_{u_o}^*, \\
&\Xi^{u_o}(y) = A(\theta)\tilde{\chi}_{u_o}^+(x), \quad y = \overline{(x,\theta)}
\end{aligned}$$

$$= s_1\tilde{u}_o^{\theta(1)} + \rho_2\tilde{u}_{u_o}^{(2)(\gamma+\theta)}$$
$$= A(\theta)\begin{cases} s_1\,\tilde{u}_o^{(1)}, & \alpha = 0, a_1 > 0,\\ s_1\tilde{u}_o^{(1)} + \rho_2\tilde{u}_o^{(2)\gamma}, & \alpha > 0,\\ \rho_2\tilde{u}_o^{(2)\gamma}, & \alpha = 0, a_1 < 0. \end{cases} \tag{6.15}$$

The map Ξ^{u_o} is well defined since

$$\begin{aligned}\Xi^{u_o}(\overline{0,\gamma,a_1,\theta}) &= \sqrt{|a_1|}\,\tilde{u}_o^{(2)(\gamma+\theta)}\\ &= \sqrt{|a_1'|}\,\tilde{u}_o^{(2)(\gamma'+\theta')}, \quad \gamma+\theta = \gamma'+\theta', a_1 = a_1' < 0\\ &= \Xi^{u_o}\big(\overline{0,\gamma',a_1',\theta'}\big).\end{aligned}$$

If we let $\alpha \to 0$, in (6.15) we get

$$\Xi^{u_o}(y) \to \begin{cases} \sqrt{a_1}\,\tilde{u}_o^{\theta}, & a_1 > 0,\\ \sqrt{|a_1|}\,\tilde{u}_o^{(2)(\gamma+\theta)}, & a_1 < 0, \end{cases}$$

and we can see that the second case is independent of $\gamma+\theta$.

Now if we let $\alpha \to 0$ in y first and then compute $\Xi^{u_o}\overline{(0,\gamma,a_1,\theta)}$ we get

$$\big\langle a_1\tilde{x}_{u_o}^{(1)} + \alpha\tilde{x}_{u_o}^{(2)\cdot\gamma}, \theta\big\rangle \to \big\langle a_1\tilde{x}_{u_o}^{(1)}, \theta\big\rangle \to \begin{cases} \sqrt{a_1}\,\tilde{u}_o^{\theta}, & a_1 > 0,\\ \sqrt{|a_1|}\,\tilde{u}_o^{(2)(\gamma+\theta)}, & a_1 < 0. \end{cases}$$

The case $a_1 < 0$ is independent of $\gamma+\theta$. This shows that Ξ^{u_o} is continuous as $\alpha \to 0$.

With this last comment we have shown that $\Psi_{u_o} : \mathbb{U}^* \to \mathfrak{X}_{u_o}^*$ is real bi-analytic with the inverse given by $\Xi_{u_o} : \mathfrak{X}_{u_o}^* \to \mathbb{U}^*$.

Appendix A

In this appendix we give some properties of the KS matrix, most of which can be shown by straightforward calculations.

A.1. Let

$$x^{(j)} = L(u)u^{(j)}, \quad j = 1,2,3,4.$$

Thus, since $u = u^{(1)}$, $x = x^{(1)}$. And

$$L(u)u = \begin{pmatrix} u_1^2 - u_2^2 - u_3^2 + u_4^2\\ 2(u_1u_2 - u_3u_4)\\ 2(u_1u_3 + u_2u_4)\\ 0 \end{pmatrix}, \qquad L(u)u^{(2)} = \begin{pmatrix} -2(u_1u_2 + 2u_3u_4)\\ u_1^2 - u_2^2 + u_3^2 - u_4^2\\ 2(u_1u_4 - u_2u_3)\\ 0 \end{pmatrix},$$

$$L(u)u^{(3)} = \begin{pmatrix} 2(-u_1u_3 + u_2u_4)\\ -2(u_1u_4 + u_2u_3)\\ u_1^2 + u_2^2 - u_3^2 - u_4^2\\ 0 \end{pmatrix}, \qquad L(u)u^{(4)} = \begin{pmatrix} 0\\ 0\\ 0\\ |u|^2 \end{pmatrix},$$

$$\|x\| = \|u\|^2,$$

$$L(u)b = \begin{pmatrix} u_1b_1 - u_2b_2 - u_3b_3 + u_4b_4 \\ u_2b_1 + u_1b_2 - u_4b_3 - u_3b_4 \\ u_3b_1 + u_4b_2 + u_1b_3 + u_2b_4 \\ u_4b_1 - u_3b_2 + u_2b_3 - u_1b_4 \end{pmatrix},$$

$$Q(u)u = \begin{pmatrix} u_1^2 - u_2^2 - u_3^2 - u_4^2 \\ 2u_1u_2 \\ 2u_1u_3 \\ 2u_1u_4 \end{pmatrix}, \qquad Q(u)b = \begin{pmatrix} u_1b_1 - u_2b_2 - u_3b_3 + u_4b_4 \\ u_2b_1 + u_1b_2 - u_4b_3 + u_3b_4 \\ u_3b_1 + u_4b_2 + u_1b_3 - u_2b_4 \\ u_4b_1 - u_3b_2 + u_2b_3 + u_1b_4 \end{pmatrix}, \tag{A.1}$$

$$Q(u) = L(u)N = \begin{pmatrix} u_1 & -u_2 & -u_3 & -u_4 \\ u_2 & u_1 & -u_4 & u_3 \\ u_3 & u_4 & u_1 & -u_2 \\ u_4 & -u_3 & u_2 & u_1 \end{pmatrix}, \qquad N = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{A.2}$$

A.2. The fibration of $\mathbb{U}^*$: Proof of Lemma 2.4. [14] Let $r = |x| = |u|^2$ for $x \in \mathbb{X}$. Pick any $u \in \psi^{-1}(x)$.

It follows from (A.1) that

$$u_1^2 + u_4^2 = \frac{r + x_1}{2}, \qquad u_2^2 + u_3^2 = \frac{r - x_1}{2}.$$

If $x_1 \geqslant 0$, pick u_1 and u_4 such that (A.1) are satisfied. Then

$$u_1^2 + u_4^2 = \frac{r + x_1}{2}, \qquad u_2 = \frac{x_2u_1 + x_3u_4}{r + x_1}, \qquad u_3 = \frac{x_3u_1 - x_2u_4}{r + x_1}. \tag{A.3}$$

For a general point in $\hat{u} \in \psi_j^{-1}(x)$ we have

$$\begin{pmatrix} \hat{u}_1 \\ \hat{u}_4 \end{pmatrix} = R(-\theta) \begin{pmatrix} u_1 \\ u_4 \end{pmatrix} \tag{A.4}$$

where

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Using (A.3), we obtain

$$\begin{pmatrix} \hat{u}_2 \\ \hat{u}_3 \end{pmatrix} = R(\theta) \begin{pmatrix} u_2 \\ u_3 \end{pmatrix}. \tag{A.5}$$

If $x_1 < 0$, pick u_2 and u_3 such that (A.1) are satisfied. Then

$$u_2^2 + u_3^2 = \frac{r - x_1}{2}, \qquad u_1 = \frac{x_2u_2 + x_3u_3}{r - x_1}, \qquad u_4 = \frac{x_3u_2 - x_2u_3}{r - x_1}. \tag{A.6}$$

In this case we let

$$\begin{pmatrix} \hat{u}_1 \\ \hat{u}_4 \end{pmatrix} = R(-\theta) \begin{pmatrix} u_1 \\ u_4 \end{pmatrix}, \qquad \begin{pmatrix} \hat{u}_2 \\ \hat{u}_3 \end{pmatrix} = R(\theta) \begin{pmatrix} u_2 \\ u_3 \end{pmatrix}. \tag{A.7}$$

In either case we have

$$\hat{u} = A(\theta)u \tag{A.8}$$

where

$$A(\theta)=\begin{pmatrix} \cos\theta & 0 & 0 & \sin\theta \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ -\sin\theta & 0 & 0 & \cos\theta \end{pmatrix} \tag{A.9}$$

which proves Lemma 2.4.

A.3. The solution set to

$$Q(u)u=x$$

is given as follows:

If $x=-a^2\mathbf{e}_1$, for some $a\neq 0$ the solution set is the 3-sphere

$$u_2^2+u_3^2+u_4^2=a.$$

Otherwise the solution set is

$$\begin{aligned} u_1^{\pm}&=\pm\sqrt{\frac{|x|+x_1}{2}},\\ u_j^{\pm}&=\pm\frac{x_j}{2u_1^{\pm}}=\pm\frac{x_j}{\sqrt{2(|x|+x_1)}}. \end{aligned} \tag{A.10}$$

Definition A.4. Let

$$I_1=\left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right),\qquad I_2=\left(\begin{array}{cc|cc} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{array}\right),$$

$$I_3=\left(\begin{array}{cc|cc} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right),\qquad I_4=\left(\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{array}\right). \tag{A.11}$$

A.5. Straightforward calculations yield the following:

$$\begin{aligned} &I_2I_3=I_4,\qquad I_3I_4=I_2,\qquad I_4I_2=I_3,\\ &I_jI_k=-I_kI_j,\quad k,j=2,3,4, && \text{(A.12)}\\ &\tau_u=-u^{(4)}, && \text{(A.13)}\\ &L\big(u^{(1)}\big)=\begin{bmatrix} u^{(1)} & u^{(2)} & u^{(3)} & u^{(4)} \end{bmatrix}=L(u),\\ &L\big(u^{(2)}\big)=\begin{bmatrix} u^{(2)} & -u^{(1)} & -u^{(4)} & u^{(3)} \end{bmatrix}=L(u)I_2,\\ &Ł\big(u^{(3)}\big)=\begin{bmatrix} u^{(3)} & u^{(4)} & -u^{(1)} & -u^{(2)} \end{bmatrix}=L(u)I_3,\\ &Ł\big(u^{(4)}\big)=\begin{bmatrix} u^{(4)} & -u^{(3)} & u^{(2)} & -u^{(1)} \end{bmatrix}=-L(u)I_4. && \text{(A.14)} \end{aligned}$$

A.6. Let

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad N = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{A.15}$$

It follows from (A.14) that

$$L(u)b = NL(b)u, \tag{A.16}$$

$$L(u)^{\mathsf{T}} b = ML(b)^{\mathsf{T}} u, \tag{A.17}$$

$$L(u)L(b) = NL(b)L(u)N, \tag{A.18}$$

$$L\big(L(u)a\big) = L(u)NL(Na), \tag{A.19}$$

$$\begin{aligned} L\big(L(u)a\big)b &= NL(b)L(u)a \\ &= L(u)L(b)Na, \end{aligned}$$

$$\begin{aligned} \frac{1}{|a|^2} L\big(L(u)a\big)Ma &= \frac{1}{|a|^2} NL(Ma)L(u)a \\ &= u, \end{aligned} \tag{A.20}$$

$$L\big(Q(u)a\big) = L(u)NL(a), \tag{A.21}$$

$$L(a)^{\mathsf{T}} NL(u) = L(u)L(Ma)N, \tag{A.22}$$

$$L(Mu) = ML(u)M, \tag{A.23}$$

$$\begin{aligned} L(u)^{\mathsf{T}} &= NL(Mu)N \\ &= NML(u)MN, \end{aligned} \tag{A.24}$$

$$Q(u)^{\mathsf{T}} = Q(Mu), \tag{A.25}$$

$$\frac{1}{|b|^2} NL(b)^{\mathsf{T}} b = \frac{1}{|b|^2} L(b)^{\mathsf{T}} b = \mathbf{b}_1, \tag{A.26}$$

$$\frac{1}{|b|^2} Q(b)^{\mathsf{T}} b = \frac{1}{|b|^2} Q(Mb)b = \frac{1}{|b|^2} NQ(Mb)b = \mathbf{b}_1, \tag{A.27}$$

$$\frac{1}{|b|^2} L(b)^{\mathsf{T}} NL(u)b = u, \tag{A.28}$$

$$\frac{1}{|b|^2} L(b)ML(u)^{\mathsf{T}} b = u. \tag{A.29}$$

Straightforward calculations show

$$Q(u) = L(u)N,$$

$$\begin{aligned} Q(u)b &= L(u)Nb \\ &= NQ(Nb)Nu \\ &= \widetilde{Q}(b)u, \end{aligned}$$

$$\widetilde{Q}(b) = \begin{pmatrix} b_1 & -b_2 & -b_3 & b_4 \\ b_2 & b_1 & b_4 & -b_3 \\ b_3 & -b_4 & b_1 & b_2 \\ b_4 & b_3 & -b_2 & b_1 \end{pmatrix}. \tag{A.30}$$

A.7. Let $R_{14}(t)$ be rotation with angle t in the $(1,4)$ direction. Similarly, let $R_{23}(t)$ be rotation with angle t in the $(2,3)$ direction.

$$\begin{aligned} A(t) &= e^{tI_4} = I\cos t + I_4 \sin t \\ &= R_{14}(-t) \oplus R_{23}(t), \end{aligned} \tag{A.31}$$

$$A(t)^{\mathsf{T}} = A(-t) = A(t)^{-1}, \tag{A.32}$$

$$\begin{aligned} &I_1 A(t) = A(t) I_1, \\ &I_2 A(t) = I_2 \cos t - I_3 \sin t = A(-t) I_2, \\ &I_3 A(t) = I_3 \cos t + I_2 \sin t = A(-t) I_3, \\ &I_4 A(t) = A(t) I_4. \end{aligned} \tag{A.33}$$

Let

$$u^{t(j)} = I_j u^t = I_j A(t) u, \quad j = 1,2,3,4.$$

Hence

$$\begin{aligned} A(t)L(u) &= \begin{bmatrix} (u^{(1)})^t & (u^{(2)})^t & (u^{(3)})^t & (u^{(4)})^t \end{bmatrix} \\ &= L_{14}(u^t) \oplus L_{23}(u^t) \\ &= L_{14}(u)R_{14}(t) \oplus L_{23}(u)R_{23}(t) \\ &= L(u)\big[R_{14}(t) \oplus R_{23}(t)\big] \\ &= L(u)B(t) \end{aligned} \tag{A.34}$$

where $L_{14}(\tilde{u}_o)$ and $L_{23}(\tilde{u}_o)$ are given by (3.3), (3.4) and

$$\begin{aligned} B(\theta) &= \begin{pmatrix} \cos\theta & 0 & 0 & -\sin\theta \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ \sin\theta & 0 & 0 & \cos\theta \end{pmatrix} \\ &= R_{14}(\theta) \oplus R_{23}(\theta). \end{aligned} \tag{A.35}$$

On the other hand

$$L(u)A(t) = L_{14}(\tilde{u}_o)R(-t) \oplus L_{23}(\tilde{u}_o)R(t). \tag{A.36}$$

It follows from (A.33) that

$$\begin{aligned} &u^{t(1)} = u^{(1)} \cos t + u^{(4)} \sin t = \big(u^{(1)}\big)^t, \\ &u^{(-s)(2)} = u^{t(2)} = u^{(2)} \cos t - u^{(3)} \sin t = \big(u^{(2)}\big)^{-t} = \big(u^{(2)}\big)^s, \\ &u^{(-s)(3)} = u^{t(3)} = u^{(2)} \sin t + u^{(3)} \cos t = \big(u^{(3)}\big)^{-t} = \big(u^{(3)}\big)^s, \\ &u^{t(4)} = -u^{(1)} \sin t + u^{(4)} \cos t = \big(u^{(4)}\big)^t. \end{aligned} \tag{A.37}$$

What about $L(u^t)$:

$$\begin{aligned} L(u^t) &= \begin{bmatrix} u^{t(1)} & u^{t(2)} & u^{t(3)} & u^{t(4)} \end{bmatrix} \\ &= \begin{bmatrix} (u^{(1)})^t & (u^{(2)})^{-t} & (u^{(3)})^{-t} & (u^{(4)})^t \end{bmatrix} \\ &= L(u)(I_1 \cos t - I_4 \sin t) \end{aligned}$$

$$
\begin{aligned}
&= L(u)A(-t) \\
&= L(u)A(t)^{-1} \\
&= L(u)A(t)^{\mathsf{T}}.
\end{aligned} \tag{A.38}
$$

Moreover,

$$
\tau_{u^t} = u^{t(4)} = u^{(4)t} = \tau_u^t, \tag{A.39}
$$

$$
\begin{aligned}
&L\big(u^{(3)}\big)u^{(3)} = L\big(u^{(2)}\big)u^{(2)} = -L(u)u, \\
&L\big(u^{(4)}\big)u^{(4)} = L(u)u = L(-u)(-u) = L(u)u.
\end{aligned} \tag{A.40}
$$

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